Numerical Constraint Solving Based on Linear Relaxations

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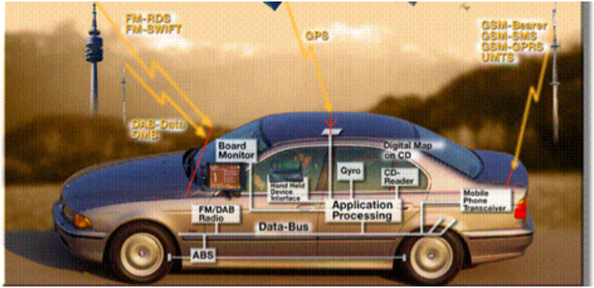
Disclaimer

Talk more of survey type

Hardly any original results

Numerical Constraints: Motivation I

By far most micro-processors nowadays do not occur in desktop PC's but embedded in technical systems (trains, cars, robots, your washing machine etc.)



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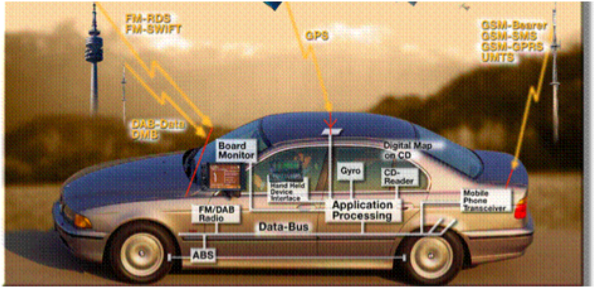

Models of technical systems usually in numerical domains.

Numerical Constraints: Motivation II

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Prototypical example: MILP

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For example:

Verification of programs with integer variables based on
invariants and ranking functions with rational/real coefficients

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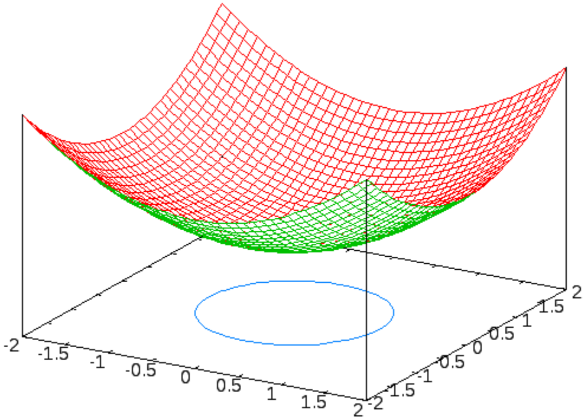

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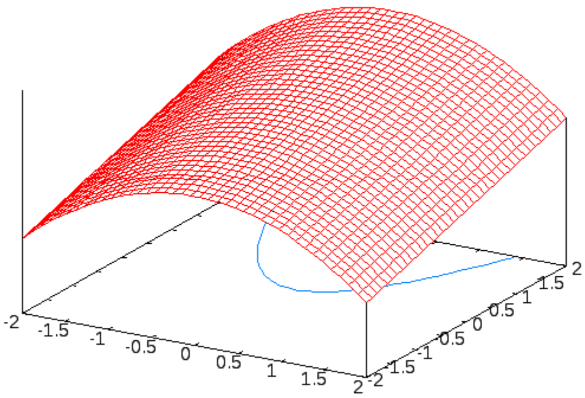

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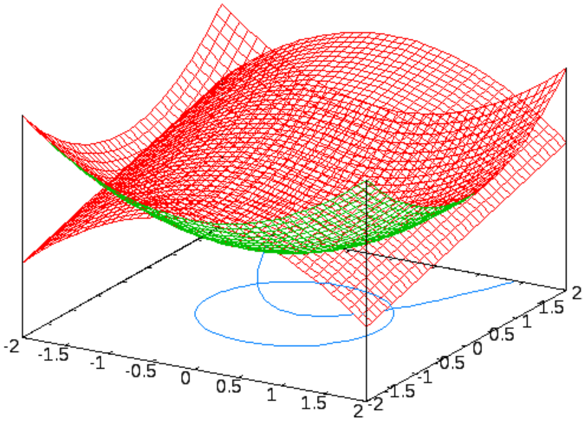

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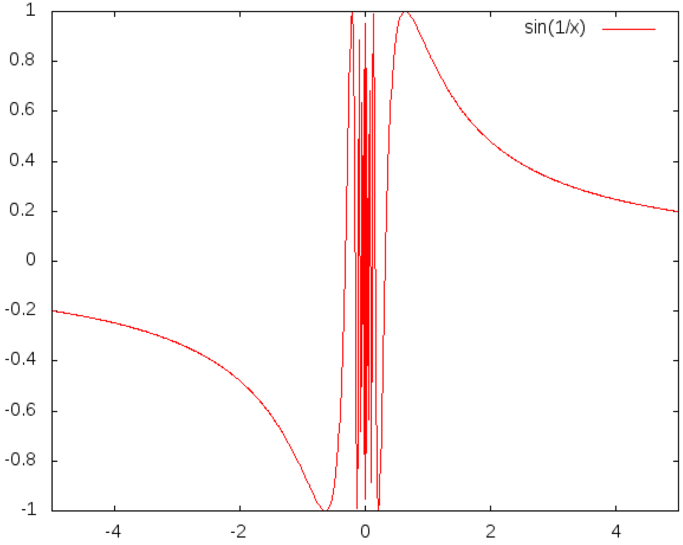

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But: We head for *quasi-decidability*:
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From now on, we assume that we search for solutions
in an n-dimensional hyper-rectangle (*box*).

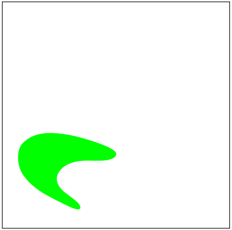
Basic Operation: Pruning

For

- a constraint ϕ in n variables
- an n-dimensional box B.

prune(ϕ, B) is a box B', such that

- $B' \subseteq B$,
- B' contains all solutions of ϕ in B:



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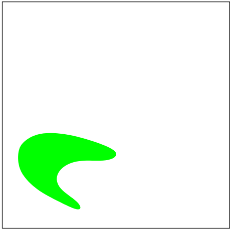

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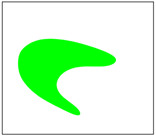

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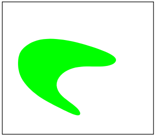

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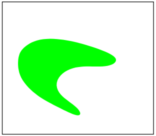

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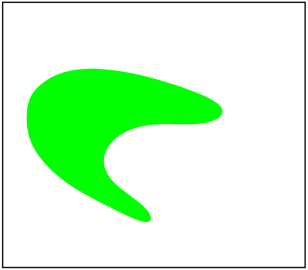
But: Usually (by design) efficient, but crude.

Branch and Prune Algorithm

Algorithm $BP(\phi, B)$:

$S \leftarrow prune(\phi, B)$
if S good enough **then** S
else
  let B_1, B_2 be s. t. $S = B_1 \cup B_2$,
    non-overlapping
  return $BP(\phi, B_1) \cup BP(\phi, B_2)$



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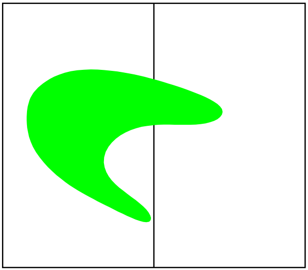

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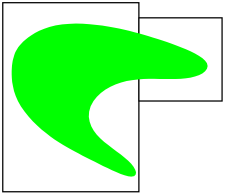

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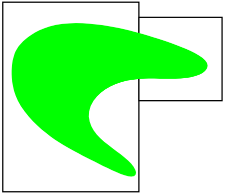

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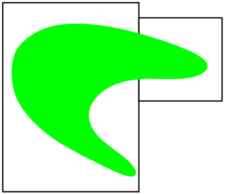

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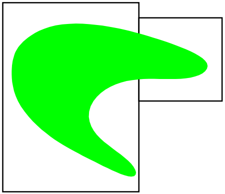

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- box small enough (size: volume, maximal side-length)
- time bound exceeded

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This search can even be built into the branch-and-prune algorithm.

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Special case: one single equality

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Input: $f = 0$, box B
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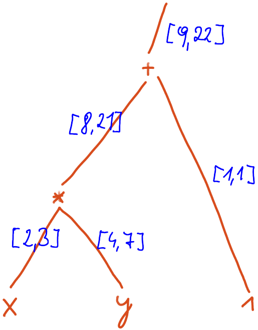
$xy + 1 = 0, x \in [2, 3], y \in [4, 7]$



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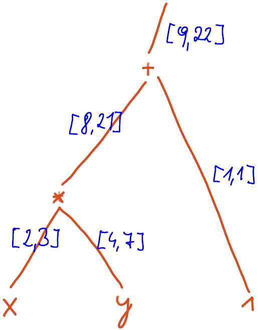

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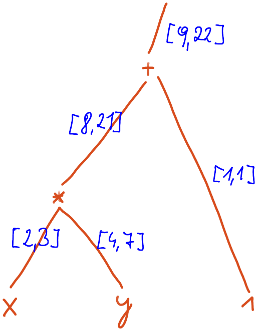

Compute interval $[f](B)$ such that $\{f(\vec{x}) \mid \vec{x} \in B\} \subseteq [f](B)$

if $0 \notin [f](I_1, \ldots, I_n)$ **then** $\emptyset$ **else** B

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- more often can live without a-priori bounds
- efficient handling of linearity
- partial results under limited time
- more scalable

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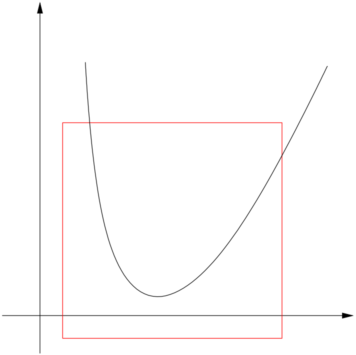
Result: linear program whose
solution set over-approximates original solution set

Linear Relaxations

$z = x + y$: already linear



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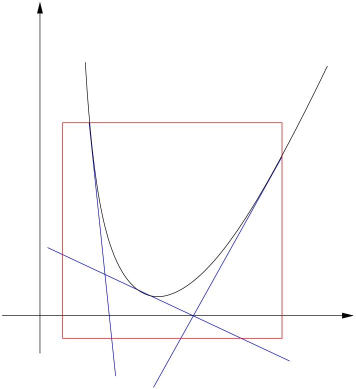

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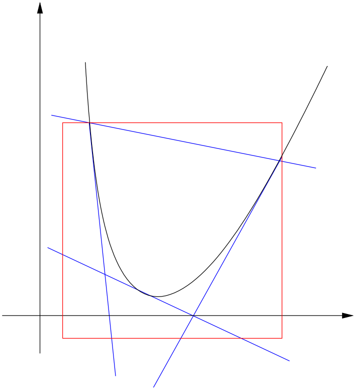

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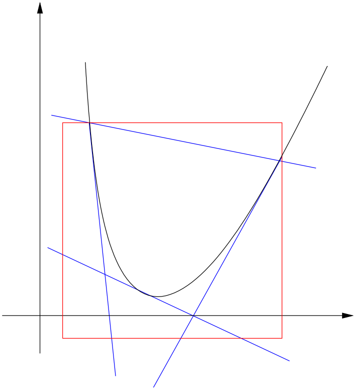

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if not convex: treat convex/concave segments separately

Linear Relaxation of Multiplication

Theorem: [McCormick, 1976]

$z = xy, x \in [\underline{x}, \overline{x}], y \in [\underline{y}, \overline{y}]$ implies

- $\underline{y}x + \underline{x}y - \underline{x}\underline{y} \leq z$
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- optimal (in general, no further implied inequalities) [Al-Khayyal and Falk, 1983]

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Moreover:

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- always at least as tight as interval arithmetic

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Here one can
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or

try to infer new variable bounds from it by solving $2n$ LPs

Implementation

We have a (very prototypical) implementation: RSolver

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`http://rsolver.sourceforge.net`

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For example, it cannot (yet) prove

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But, in theory [Franek et al., 2010] we can already do this, too.

Conclusion

Constraint solvers for the real numbers,
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can be useful for analyzing discrete problem.

If you want to try, contact us.

Literature I

Faiz A. Al-Khayyal and James E. Falk. Jointly constrained biconvex programming. *Mathematics of Operations Research*, 8 (2):273–286, 1983.

Peter Franek, Stefan Ratschan, and Piotr Zgliczynski. Satisfiability of systems of equations of real analytic functions is quasi-decidable. `http://www.cs.cas.cz/~ratschan/preprints.html`, 2010.

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Literature II

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