Automata on Infinite Trees

Infinite Binary Trees

We consider the *infinite complete binary tree* $t : \{0,1\}^* \rightarrow \Sigma$ over an unranked alphabet $\Sigma = \{a, b, c, \ldots\}$.

Let $t : D \rightarrow \Sigma$, where $D \subseteq \mathbb{N}^*$ is a prefix-closed set be a k-ary tree.

We encode t as an infinite binary tree $T : \{0,1\}^* \rightarrow \Sigma_\perp$, where:

- for all positions $n_1 n_2 \ldots n_p \in D$, $n_i < k$, we have

$$T(1^{n_1}01^{n_2}0\ldots 1^{n_p}) = t(n_1 n_2 \ldots n_p)$$

- $T(x) = \perp$ if $x \notin \{1^{n_1}01^{n_2}0\ldots 1^{n_p} \mid n_1 n_2 \ldots n_p \in D\}$.

Büchi Automata on Infinite Trees

Definition

A Büchi tree automaton over Σ is $A = \langle S, I, T, F \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is the *transition relation*,
- $F \subseteq S$ is the set of *final states*.

Runs

A *run* of A over a tree $t : \{0,1\}^* \longrightarrow \Sigma$ is a mapping $\pi : \{0,1\}^* \longrightarrow S$ such that, for each position $p \in \{0,1\}^*$, where $q = \pi(p)$, we have:

- if $p = \epsilon$ then $q \in I$, and
- if $q_i = \pi(pi)$, $i = 0, 1$ then $\langle q_0, q_1 \rangle \in T(q, t(p))$.

If π is a run of A and σ is a path in t, let $\pi_{|\sigma}$ denote the path in π corresponding to σ.

A run π is said to be *accepting*, if and only if for every path σ in t we have:

$$\inf(\pi_{|\sigma}) \cap F \neq \emptyset$$

Closure Properties

For every Büchi automaton A there exists a complete Büchi automaton A' such that $\mathcal{L}(A) = \mathcal{L}(A')$.

Theorem 1 *The class of Büchi-recognizable tree languages is closed under union, intersection and projection.*

Let $A_i = \langle S_i, I_i, T_i, F_i \rangle$, $i = 1, 2$, where $S_1 \cap S_2 = \emptyset$.

Let $A_\cup = \langle S_1 \cup S_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2 \rangle$.

Closure Properties

Let $A_\cap = \langle S, I, T, F \rangle$ where:

- $S = S_1 \times S_2 \times \{0, 1, 2\}$
- $I = I_1 \times I_2 \times \{1\}$
- for any $s, s_1, s_2 \in S_1$, $s', s'_1, s'_2 \in S_2$, $a, b \in \{0, 1, 2\}$:

$$\langle (s_1, s'_1, b), (s_2, s'_2, b) \rangle \in T((s, s', a), \sigma)$$

  iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$, $\langle s'_1, s'_2 \rangle \in T(s', \sigma)$ and:

  1. if $a = 0$ or ($a = 1$ and $s \notin F_1$), then $b = 1$
  2. if ($a = 1$ and $s \in F_1$) or ($a = 2$ and $s \notin F_1$), then $b = 2$
  3. if $a = 2$ and $s' \in F_2$, then $b = 0$

- $F = S \times S \times \{0\}$

Emptiness of Büchi Automata

Let $A = \langle S, I, T, F \rangle$ be a Büchi tree automaton where $F = \{s_1, \ldots, s_m\}$, and $\pi : \{0,1\}^* \to S$ be a successful run of A on the tree $t \in \mathcal{T}(\Sigma)$.

For any $s \in S$, and any $u \in \{0,1\}^*$, let

$$d_u^\pi = \{w \in u \cdot \{0,1\}^* \mid \pi(v) \notin F, \text{ for all } u < v < w\}$$

By König's lemma, d_u^π is finite for any $u \in \{0,1\}^*$.

If $\pi(u) = s$, let t_s^π be the restriction of t to d_u^π. Let

$$T_s = \{t_s^\pi \mid \pi \text{ is a successful run of } A \text{ on } t\}$$

Emptiness of Büchi Automata

If $\vec{s} = \langle s_1, \ldots, s_m \rangle$:

$$\mathcal{L}(A) = \bigcup_{s_0 \in I} T_{s_0} \cdot_{\vec{s}} \langle T_{s_1}, \ldots, T_{s_m} \rangle^{\omega \vec{s}}$$

Conversely, the expression above denotes a Büchi-recognizable tree language.

Let $A = \langle S, I, T, F \rangle$ be a Büchi tree automaton. For each $s \in S$ let T_s be the rational tree language defined above. Eliminate from S (and T) all states s such that $T_s = \emptyset$, and let S' be the resulting set of states.

We claim that $\mathcal{L}(A) \neq \emptyset \iff S' \cap I \neq \emptyset$.

The Complement Problem

Let $\Sigma = \{a, b\}$, $\mathcal{T}_0 = \{t \in \mathcal{T}^\omega(\Sigma) \mid \text{some path in } t \text{ has infinitely many } a\text{'s}\}$

$\mathcal{T}_0$ is Büchi recognizable.

Let $A = \langle \{s_0, s_1, s_a, s_b\}, \{s_0\}, T, \{s_1, s_a\} \rangle$, where T is defined by:

$$\begin{aligned}
a(s_{0,a,b}) &\rightarrow \{\langle s_1, s_a \rangle, \langle s_a, s_1 \rangle\} \\
b(s_{0,a,b}) &\rightarrow \{\langle s_1, s_b \rangle, \langle s_b, s_1 \rangle\} \\
a(s_1) &\rightarrow \{\langle s_1, s_1 \rangle\} \\
b(s_1) &\rightarrow \{\langle s_1, s_1 \rangle\}
\end{aligned}$$

The Complement Problem

Let $\mathcal{T}_1 = \mathcal{T}^\omega(\Sigma) \setminus \mathcal{T}_0 = \{t \in \mathcal{T}^\omega(\Sigma) \mid \text{all paths in } t \text{ have finitely many } a\text{'s}\}$.

We show that $\mathcal{T}_1$ cannot be recognized by a Büchi tree automaton.

Let $T_n : \{0,1\}^* \to \Sigma$ be the language of trees

$$t_n(p) = \begin{cases} \text{a} & \text{if } p \in \{\epsilon, 1^{m_1}0, 1^{m_1}01^{m_2}0, \ldots, 1^{m_1}01^{m_2}0 \ldots 1^{m_n}0 \mid m_1, \ldots m_n \in \mathbb{N}\} \\ \text{b} & \text{otherwise} \end{cases}$$

Obviously, $T_n \subset \mathcal{T}_1$, for all $n \in \mathbb{N}$.

The Complement Problem

We show, by induction on $n \geq 2$, that, for all Büchi automata A with $n-1$ states, and all $t_m \in \mathcal{L}(A) \cap T_m$, with $1 \leq m \leq n$:

- there exists a path σ in t_m and $u < v < w < \sigma$, such that $\pi(u) = \pi(w) = s \in F$ and $t_m(v) = a$.

Then $\pi = r_1 \cdot_s r_2 \cdot_s r_3$, and $r_1 \cdot_s r_2^{\omega s}$ is a successful run on $q_1 \cdot q_2^{\omega}$, which contains a path with infinitely many a.

Müller Automata on Infinite Trees

Definition

A Müller tree automaton Σ is $A = \langle S, I, T, \mathcal{F} \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is the *transition function*,
- $\mathcal{F} \subseteq 2^S$, is the set of *accepting sets*.

A run π of A over t is said to be *accepting*, iff for every path σ in t:

$$\inf(\pi_{|\sigma}) \in \mathcal{F}$$

Closure Properties

The class of Müller-recognizable tree languages is closed under union and intersection.

For union, the proof is exactly as in the case of Büchi automata. For $A_\cup$, the set of accepting sets is the union of the sets $\mathcal{F}_i$, $i = 1, 2$.

For intersection, let $A_\cap = \langle S_1 \times S_2, I_1 \times I_2, T, \mathcal{F} \rangle$, where:

- $\langle (s_1, s_1'), (s_2, s_2') \rangle \in T((s, s'), \sigma)$ iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$ and $\langle s_1', s_2' \rangle \in T(s', \sigma)$, and
- $\mathcal{F} = \{G \in S_1 \times S_2 \mid pr_1(G) \in \mathcal{F}_1 \text{ and } pr_2(G) \in \mathcal{F}_2\}$, where:
  - $pr_1(G) = \{s \in S_1 \mid \exists s' \,.\, (s, s') \in G\}$, and
  - $pr_2(G) = \{s \in S_2 \mid \exists s' \,.\, (s', s) \in G\}$.

Rabin Automata on Infinite Trees

Definition

A Rabin tree automaton Σ is $A = \langle S, I, T, \Omega \rangle$, where:

- S is a finite set of *states*,
- $I \subseteq S$ is a set of *initial states*,
- $T : S \times \Sigma \rightarrow 2^{S \times S}$ is the *transition function*,
- $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle P_n, N_n \rangle\}$ is the set of *accepting pairs*.

A run π of A over t is said to be *accepting*, if and only if for every path σ in t there exists a pair $\langle N_i, P_i \rangle \in \Omega$ such that:

$$\inf(\pi_{|\sigma}) \cap N_i = \emptyset \text{ and } \inf(\pi_{|\sigma}) \cap P_i \neq \emptyset$$

Büchi, Müller and Rabin

For every Büchi tree automaton A there exists a Rabin tree automaton B, such that $\mathcal{L}(A) = \mathcal{L}(B)$, but not viceversa.

For every Müller tree automaton A there exists a Rabin tree automaton B, such that $\mathcal{L}(A) = \mathcal{L}(B)$, and viceversa.

The Rabin Complementation Theorem

Theorem 2 (Rabin '69) *The class of Rabin-recognizable tree languages is closed under complement.*

The class of Rabin-recognizable tree languages is closed under union and intersection.

Emptiness of Rabin Automata

Given an alphabet Σ, an infinite tree $t \in \mathcal{T}^\omega(\Sigma)$ is said to be *regular* if there are only finitely many distinct subtrees t_u of t, where $u \in \{0,1\}^*$.

Example 1 *The infinite binary tree* $f(g(f(\ldots), f(\ldots)), g(f(\ldots), f(\ldots)))$ *is regular.* $\square$

Theorem 3 (Rabin '72)

1. *Any non-empty Rabin-recognizable set of trees contains a regular tree.*
2. *The emptiness problem for Rabin tree automata is decidable.*

Reduction to empty alphabet

Let $A = \langle S, I, T, \Omega \rangle$ be a Rabin tree automaton over Σ, such that $\mathcal{L}(A) \neq \emptyset$, where $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_n, P_n \rangle\}$.

Let $A' = \langle S \times \Sigma, I \times \Sigma, T', \Omega' \rangle$, where:

- $\langle (s_1, \sigma_1), (s_2, \sigma_2) \rangle \in T'((s, \sigma))$ iff $\langle s_1, s_2 \rangle \in T(s, \sigma)$, and $\sigma_1, \sigma_2 \in \Sigma$.
- $\Omega' = \{\langle N_1 \times \Sigma, P_1 \times \Sigma \rangle, \ldots, \langle N_n \times \Sigma, P_n \times \Sigma \rangle\}$.

The successful runs of A' are pairs (π, t), where $t \in \mathcal{L}(A)$, and π is a successful run of A on t.

Regular successful runs

For any Rabin tree automaton A, there exists a Rabin tree automaton A' with one initial state such that $\mathcal{L}(A) = \mathcal{L}(A')$.

Consider a Rabin tree automaton $A = \langle S, s_0, T, \Omega \rangle$ over the empty alphabet, and let π be a successful run of A.

Claim 1 *If A has a successful run, A has also a regular successful run.*

A state $s \in S$ is said to be *live* if $s \neq s_0$ and $\langle s_1, s_2 \rangle \in T(s)$ for some $s_1, s_2 \in S$, where either $s_1 \neq s$ or $s_2 \neq s$.

By induction on $n =$ the number of live states in A.

Regular successful runs

If $n = 0$, $\pi(\epsilon) = s_0$ and $\pi(p) = s$, for all $p \in dom(\pi)$, and $s \in S$ non-live.

Case 1 If some live state in A is missing on π, apply the induction hypothesis.

Case 2 All states of A appear on π, and there is a position $u \in \{0, 1\}^*$ such that $\pi(u) = s$ is live, but some live state s' does not appear in π_u.

Let $\pi_1 = \pi \setminus \pi_u$ and $\pi_2 = \pi_u$. Both π_1 and π_2 are runs of automata with $n - 1$ live states, hence there exists successful regular runs π'_1 and π'_2 of these automata. The desired run is $\pi'_1 \cdot_s \pi'_2$.

Regular successful runs

Case 3 All live states appear in any subtree of π. Let σ be a path in π consisting of all the live states appearing again and again, and only of the live states, with the exception of $\pi(\lambda)$. **Q:** Why does σ exist?

There exists $\langle N, P\rangle \in \Omega$, such that $\inf(\sigma) \cap N = \emptyset$ and $\inf(\sigma) \cap P \neq \emptyset$. Then N contains only non-live states.

Let $s \in \inf(\sigma) \cap P$ and u, v be the 1^{st} and 2^{nd} positions such that $\sigma(u) = \sigma(v) = s$.

Let $\pi_1 = \pi \setminus \pi_u$ and $\pi_2 = \pi_u \setminus \pi_v$. Both π_1 and π_2 are runs of automata with $n-1$ live states, hence there exists successful regular runs π_1' and π_2' of these automata. The desired run is $\pi_1' \cdot_s {\pi_2'}^{\omega s}$.

The Emptiness Problem

Let A be an input-free Rabin tree automaton with n live states.

We derive $A_{n-1}, A_{n-2}, \ldots, A_0$ from A, having $n-1, n-2, \ldots 0$ live states.

If A has a successful run, then it it has a regular run, composed of runs of $A_{n-1}, A_{n-2}, \ldots, A_0$.

So it is enough to check emptiness of $A_{n-1}, A_{n-2}, \ldots, A_0$.

Rabin Automata, SkS and SωS

Defining infinite paths

We say that a set of positions X is linear iff the following holds:

$$linear(X) \ : \ (\forall x, y \ . \ X(x) \wedge X(y) \rightarrow x \leq y \vee y \leq x)$$

X is a path iff:

$$path(X) \ : \ linear(X) \wedge \forall Y \ . \ linear(Y) \wedge X \subseteq Y \rightarrow X = Y$$

From Automata to Formulae

Let $A = \langle S, I, T, \Omega \rangle$ be a Rabin tree automaton, where $S = \{s_1, \ldots, s_p\}$.

Let $\vec{Y} = \{Y_1, \ldots, Y_p\}$ be set variables.

If X denotes a path, state i appears infinitely often in X iff:

$$inf_i(X) \ : \ \forall x \ . \ X(x) \rightarrow \exists y \ . \ x \leq y \wedge X(y) \wedge Y_i(y)$$

The formula expressing the accepting condition is:

$$\Phi_\Omega(\vec{Y}) \ : \ \forall X \ . \ path(X) \rightarrow \bigvee_{\langle N,P \rangle \in \Omega} \Big(\bigwedge_{s_i \in N} \neg inf_i(X) \ \wedge \bigvee_{s_i \in P} inf_i(X) \Big)$$

Decidability of S2S

Theorem 4 *Given an alphabet* Σ, *a tree language* $L \subseteq \mathcal{T}^\omega(\Sigma)$ *is definable in S2S iff it is rational.*

Corollary 1 *The SAT problem for S2S is decidable.*

Obtaining Decidability Results by Reduction

Suppose we have a logic $\mathcal{L}$ interpreted over the domain $\mathcal{D}$, such that the following problem is decidable:

for each formula φ of $\mathcal{L}$ there exists $\mathfrak{m} \in \mathcal{D}$ such that $\mathfrak{m} \models \varphi$

Then we can prove the same thing for another logic $\mathcal{L}'$ interpreted over $\mathcal{D}'$ iff there exists functions $\Delta : \mathcal{D}' \to \mathcal{D}$ and $\Lambda : \mathcal{L}' \to \mathcal{L}$ such that for all $\mathfrak{m}' \in D'$ and $\varphi' \in \mathcal{L}$ we have:

$$\mathfrak{m}' \models \varphi' \iff \Delta(\mathfrak{m}') \models \Lambda(\varphi')$$

Decidability of SωS

Every tree $t : \mathbb{N}^* \to \Sigma$ can be encoded as $t' : \{0,1\}^* \to \Sigma$.

Let $D = \{\epsilon\} \cup \bigcup_{n_1,\ldots,n_k \in \mathbb{N}} \{1^{n_1}01^{n_2}0\ldots 1^{n_k}0 \mid k \geq 1, n_i \geq 1, 1 \leq i \leq k\}$.

Embedding the domain of SωS into S2S:

$$
\begin{aligned}
D(x) \quad : \quad & \exists z \forall y \,.\, (z \leq y) \wedge x = z \ \vee \\
& \Big(s_1(z) \leq x \wedge \exists y \,.\, z < y \wedge s_0(y) = x \wedge \\
& \forall y \,.\, \big(s_0(y) < x \to s_1(s_0(y)) < x\big)\Big)
\end{aligned}
$$

Decidability of SωS

If $p = 1^{n_1}01^{n_2}0\ldots1^{n_k}0$, let $f_i(p) = 1^{n_1}01^{n_2}0\ldots1^{n_k}01^i0$

$$x \preceq_1 y \quad : \quad D(x) \wedge D(y) \wedge x \preceq y$$

Define the relation $x \leq_D y$ iff $x \in D$ and $y = x1^n0$, for some $n \in \mathbb{N}$.

Define $f_0, f_1, f_2, \ldots$ by induction:

- $f_0(x) = y \;:\; D(x) \wedge D(y) \wedge x \leq_D y \wedge \forall z \,.\, x \leq_D z \rightarrow y \preceq_1 z$
- $f_{i+1}(x) = y \;:\; D(x) \wedge D(y) \wedge x \leq_D y \wedge \forall z \,.\, x \leq_D z \rightarrow y \preceq_1 z \wedge \bigwedge_{0 \leq k \leq i} y \neq f_k(x)$.