Obligation and Weak-Parity Games

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LTL Hierarchy

Reactivity

Recurrence/Büchi  Persistence/co-Büchi

Obligation

Safety  Reachability/Guarantee
Obligation Games

We consider games where the winning condition for Player 0 (on the play) is

- a Boolean combination of reachability conditions
- equivalently: a condition on the set Occ

Standard form: Staiger-Wagner winning condition, using

\[ F = \{F_1, \ldots, F_k\} \]

Player 0 wins play \( \rho \) iff \( \text{Occ}(\rho) \in F \). We call these games obligation games (or Staiger-Wagner games).
Example

\[ S = \{s_1, s_2, s_3\} \quad F = \{\{s_1, s_2, s_3\}\} \]

No winning strategy is positional.

There is a finite-state winning strategy.
Weak Parity Games

Method for solving Staiger-Wagner games:

1. Solve weak parity games.

2. Reduce Staiger-Wagner games to weak parity games.

A weak parity game is a pair \((G, p)\), where

\begin{itemize}
  \item \(G = (S, S_0, E)\) is a game graph and
  \item \(p : S \rightarrow \{0, \ldots, k\}\) is a priority function mapping every state in \(S\) to a number in \(\{0, \ldots, k\}\).
\end{itemize}

A play \(\rho\) is winning for Player 0 iff the minimum priority occurring in \(\rho\) is even: \(\min_{s \in \text{Occ} (\rho)} p(s)\) is even
Example
Weak Parity Games

Theorem

For a weak parity game one can compute the winning regions $W_0$, $W_1$ and also construct corresponding positional winning strategies.

Proof.

Let $G = (S, S_0, E)$ be a game graph, $p : S \to \{0, \ldots, k\}$ a priority function. Let $P_i = \{s \in S \mid p(s) = i\}$.

First steps if $P_0 \neq \emptyset$: We first compute $A_0 = \text{Attr}_0(P_0)$, clearly from here Player 0 can win.

In the rest game, we compute $A_1 = \text{Attr}_1(P_1 \setminus A_0)$ from here Player 1 can win.
General Construction

**Aim:** Compute $A_0, A_1, \ldots A_k$

Let $G_i$ be the game graph restricted to $S \setminus (A_0 \cup \ldots A_{i-1})$.

$\text{Attr}^{G_i}_0(M)$ is the $0$-attractor of $M$ in the subgraph induced by $G_i$.

$A_0 := \text{Attr}_0(P_0)$

$A_1 := \text{Attr}^{G_1}_1(A_0 \setminus P_1)$

for $i > 1$:

$A_i := \begin{cases} 
\text{Attr}^{G_i}_0(P_i \setminus (A_0 \cup \ldots \cup A_{i-1})) & \text{if } i \text{ is even} \\
\text{Attr}^{G_i}_1(P_i \setminus (A_0 \cup \ldots \cup A_{i-1})) & \text{if } i \text{ is odd}
\end{cases}$
Correctness

Correctness Claim:

\[ W_0 = \bigcup_{i \text{ even}} A_i \quad \text{and} \quad W_1 = \bigcup_{i \text{ odd}} A_i \]

and the union of the corresponding attractor strategies are positional winning strategies for the two players on their respective winning regions.

Prove by induction on \( j = 0, \ldots, k \) the following:

\[ \bigcup_{i=0..k, i \text{ even}} A_i \subseteq W_0 \quad \text{and} \quad \bigcup_{i=1..k, i \text{ odd}} A_i \subseteq W_0 \]
Correctness (cont.)

Base:

- i=0: \( A_0 = Attr_0(P_0) \subseteq W_0 \)
- i=1: \( A_1 = Attr_1(P_1 \setminus A_0) \subseteq W_1 \)

Induction step:

- i even: Consider play \( \rho \) starting \( A_i \) that complies to attractor strategy.
  - Case 1: \( \rho \) eventually leaves \( A_i \) to some \( A_j \) (from a Player-1 state), which \( j < i \) and even, then Player 0 wins by induction hypothesis.
  - Case 2: \( \rho \) visits \( P_i \), then we need to show that \( \rho \) visits only states with \( p(s) \geq i \). Consider a state \( s \) that visits \( P_i \), then
    - if \( s \in S_0 \), then not all edges lead to states with lower priority, otherwise \( s \in A_j \) for some \( j < i \). Contradiction.
Correctness (cont.)

- Case 2 (cont.):
  - if $s \in S_1$, then all edges lead to states with priority $\geq i$. Any edge to a lower priority must lead to $A_j$ with even $j$ (Case 1). If there were edges to states $s'$ with priority $j < i$ and $j$ odd, then $s'$ would already be in $A_j$. Contradiction.

- $i$ odd: switch players
How to translate a Staiger-Wagner automaton to Weak-Parity automaton?

Idea: record visited states during a run

Record set: $R \subseteq S$

Question: How to give priorities?
Record Sets and Priorities

Assume automaton with states \(\{s_0, s_1, s_2\}\). Consider possible record sets.

Assume the following run \(s_1, s_0, s_1, s_0, s_2, \ldots\) and the acceptance condition \(F = \{\{s_0, s1\}, \{s0, s1, s2\}\}\). How to assign priorities?
Record Sets and Priorities

\( F = \{\{s_0, s_1\}, \{s_0, s_1, s_2\}\} \). How would you assign priorities?

\[
\begin{align*}
\{s_0, s_1, s_2\} &: 0 \\
\{s_0, s_1\} &: 2 \quad \{s_0, s_2\} &: 3 \\
\{s_0\} &: 5 \\
\emptyset &: \text{d.c.}
\end{align*}
\]
From Staiger-Wagner to Weak Parity Automata

Given a deterministic Staiger-Wagner automaton $A = (S, I, T, F)$, we can construct an equivalent weak parity automaton $A' = (S', I', T', p)$ as follows:

\[
S' := S \times 2^S \\
I' := (I, \{I\}) \\
T'(\langle s, R \rangle, a) := (T(s, a), R \cup \{T(s, a)\}) \\
p((s, R)) := 2 \cdot |S| - \begin{cases} 
2 \cdot |R| & \text{if } R \in F \\
2 \cdot |R| - 1 & \text{if } R \not\in F
\end{cases}
\]
Idea of Game Reduction

We want to solve Staiger-Wagner games. We use a reduction to weak parity games (and the positional winning strategies of weak parity games).

Reduction will transform a game \((G, \phi)\) into a game \((G', \phi')\) such that usually

- \(G'\) is (usually) larger than \(G\)
- \(\phi'\) is simpler than \(\phi\) (so the solution of \((G', \phi')\) is simpler than that of \((G, \phi)\))
- from a solution of \((G', \phi')\) we can construct a solution of \((G, \phi)\).

Concrete application: Transform Staiger-Wagner game into a weak parity game over a larger graph (from \(S\) proceed to \(S \times 2^S\))
Game Reduction

Let $G = (S, S_0, E)$ and $G' = (S', S'_0, E')$ be game graphs with winning conditions $\phi$ and $\phi'$, respectively.

$(G, \phi)$ is reducible to $(G', \phi')$ if:

1. $S' = S \times M$ for a finite set $M$ and $S'_0 = S_0 \times M$

2. Each play $\rho = s_0s_1\ldots$ over $G$ is translated into a play $\rho' = s'_0s'_1\ldots$ over $G'$ by
   - a function $f : S \to S \times M$ (the beginning of $\rho'$).
   - forall states $(m, s) \in S \times M$ in $G'$ and all states $s' \in S$ in $G$, if there exists an edge $(s, s') \in E$, then there is a unique $m'$ with $((m, s), (m', s')) \in E'$
   - forall edges $((m, s), (m', s')) \in E'$ in $G'$, there is an edges $(s, s') \in E$ in $G$

3. For all plays $\rho$ and $\rho'$ according to 2.: $\rho \in \phi$ iff $\rho' \in \phi'$
Application of Game Reduction

Theorem

Suppose \((G, \phi)\) is reducible to \((G', \phi')\) with extension set \(M\), initial function \(g\), and \(G\) and \(G'\) defined as before. Then, if Player 0 wins in \((G', \phi')\) from \(g(s)\) with a memoryless winning strategy, then Player 0 wins in \((G, \phi)\) from \(s\) with a finite-state strategy.

Idea: Given a memoryless winning strategy \(f : S'_0 \rightarrow S'\) from \(g(s)\) for Player 0 in \((G', \phi')\), we can construct a strategy automaton \(A = (M, m_0, \delta, \lambda)\) for Player 0 in \((G, \phi)\).
Obligation/Staiger-Wagner Games

Theorem

Given a Staiger-Wagner game \((G, \phi)\), one can compute the winning regions of Player 0 and 1 and corresponding finite state strategies.

Proof.

We can apply game reduction with \((G', \phi')\) as follows:

\[
G' := (S', S'_0, E')
\]

\[
S' := 2^S \times S
\]

\[
((R, s), (R', s')) \in E' \iff (s, s') \in E, R' = R \cup \{s'\}
\]

\[
g(s) = (\{s\}, s)
\]

\[
p((R, s)) := 2 \cdot |S| - \begin{cases} 2 \cdot |R| & \text{if } P \in \phi \\ 2 \cdot |R| - 1 & \text{if } P \notin \phi \end{cases}
\]
Exponential-Size Memory

Theorem

There is a family of Staiger-Wagner games over game graphs \( G_1, G_2, G_3, \ldots \) which grow linearly in \( n \) such that

- Player 0 wins from a certain initial vertex of \( G_n \)
- any finite-state strategy for Player 0 needs at least \( 2^n \) states

Winning condition:

\[
\phi = \{ \rho \mid \forall i = 1 \ldots n : i \in \text{Occ}(\rho) \iff i' \in \text{Occ}(\rho) \}
\]
Exponential Memory (cont.)

Claim:
Over $G_n$ there is an automaton winning strategy for Player 0 from vertex $s_0$ with a memory of size $2^n$. (Remember the visited vertices $i$, for the appropriate choice from vertex $s'_0$ onwards.)
Each automaton winning strategy for Player 0 from $s_0$ in $G_n$ has a memory of $2^n$ many states.

Proof.
Assume $|\text{states}| < 2^n$ is sufficient.
Then two play prefixes $u \neq v$ exist leading to the same memory states at $s'_0$. The rest $r$ of the play is then the same after $u$ and $v$.
One of the two player $ur$, $vr$ is lost by Player 0. Contradiction.