Büchi Automata
Definition of Büchi Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

By $\Sigma^\omega$ we denote the set of all infinite words over $\Sigma$.

A non-deterministic Büchi automaton (NBA) over $\Sigma$ is a tuple $A = \langle S, I, T, F \rangle$, where:

- $S$ is a finite set of states,
- $I \subseteq S$ is a set of initial states,
- $T \subseteq S \times \Sigma \times S$ is a transition relation,
- $F \subseteq S$ is a set of final states.
Acceptance Condition

A run of a Büchi automaton is defined over an infinite word $w : \alpha_1 \alpha_2 \ldots$ as an infinite sequence of states $\pi : s_0 s_1 s_2 \ldots$ such that:

- $s_0 \in I$ and
- $(s_i, \alpha_{i+1}, s_{i+1}) \in T$, for all $i \in \mathbb{N}$.

$$\text{inf}(\pi) = \{s \mid s \text{ appears infinitely often on } \pi\}$$

Run $\pi$ of $A$ is said to be accepting iff $\text{inf}(\pi) \cap F \neq \emptyset$. 
Examples

Let $\Sigma = \{0, 1\}$. Define Büchi automata for the following languages:

1. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs in } \alpha \text{ exactly once}\}$
2. $L = \{\alpha \in \Sigma^\omega \mid \text{after each } 0 \text{ in } \alpha \text{ there is } 1\}$
3. $L = \{\alpha \in \Sigma^\omega \mid \alpha \text{ contains finitely many } 1\text{'s}\}$
4. $L = (01)^*\Sigma^\omega$
5. $L = \{\alpha \in \Sigma^\omega \mid 0 \text{ occurs on all even positions in } \alpha\}$
Closure Properties

Closure under union and projection are like in the finite automata case.

Intersection is a bit special.

Complementation of non-deterministic Büchi automata is a complex result.

Deterministic Büchi automata are not closed under complement.
Closure under Intersection

Let $A_1 = \langle S_1, I_1, T_1, F_1 \rangle$ and $A_2 = \langle S_2, I_2, T_2, F_2 \rangle$

Build $A_\cap = \langle S, I, T, F \rangle$:

- $S = S_1 \times S_2 \times \{1, 2, 3\}$,
- $I = I_1 \times I_2 \times \{1\}$,
- the definition of $T$ is the following:
  - $((s_1, s_1', 1), a, (s_2, s_2', 1)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1 \notin F_1$
  - $((s_1, s_1', 1), a, (s_2, s_2', 2)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1 \in F_1$
  - $((s_1, s_1', 2), a, (s_2, s_2', 2)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1' \notin F_2$
  - $((s_1, s_1', 2), a, (s_2, s_2', 3)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$ and $s_1' \in F_2$
  - $((s_1, s_1', 3), a, (s_2, s_2', 1)) \in T$ iff $(s_i, a, s_i') \in T_i$, $i = 1, 2$
- $F = S_1 \times S_2 \times \{3\}$
The Emptiness Problem

Theorem 1  Given a Büchi automaton $A$, $\mathcal{L}(A) \neq \emptyset$ iff there exist $u, v \in \Sigma^*$, $|u|, |v| \leq \|A\|$, such that $uv^\omega \in \mathcal{L}(A)$.

In practical terms, $A$ is non-empty iff there exists a state $s$ which is reachable both from an initial state and from itself.

Q: Is the membership problem decidable for Büchi automata?
Complementation of Büchi Automata
**Congruences**

**Definition 1** An equivalence relation \( R \subseteq \Sigma^* \times \Sigma^* \) is said to be a **left-congruence** iff for all \( u, v, w \in \Sigma^* \) we have \( u \equiv v \Rightarrow wu \equiv vw \).

**Definition 2** An equivalence relation \( R \subseteq \Sigma^* \times \Sigma^* \) is said to be a **right-congruence** iff for all \( u, v, w \in \Sigma^* \) we have \( u R v \Rightarrow uw R vw \).

**Definition 3** An equivalence relation \( R \subseteq \Sigma^* \times \Sigma^* \) is said to be a **congruence** iff it is both a left- and a right-congruence.

Ex: the Myhill-Nerode equivalence \( \sim_L \) is a right-congruence.
**Congruences**

Let $A = \langle S, I, T, F \rangle$ be a Büchi automaton and $s, s' \in S$.

$$W_{s,s'} = \{ w \in \Sigma^* \mid s \xrightarrow{w} s' \}$$

For $s, s' \in S$ and $w \in \Sigma^*$, we denote $s \xrightarrow{w}_F s'$ iff $s \xrightarrow{w} s'$ visiting a state from $F$.

$$W_{s,s'}^F = \{ w \in \Sigma^* \mid s \xrightarrow{w}_F s' \}$$

For any two words $u, v \in \Sigma^*$ we have $u \cong v$ iff for all $s, s' \in S$ we have:

- $s \xrightarrow{u} s' \iff s \xrightarrow{v} s'$, and
- $s \xrightarrow{F} s' \iff s \xrightarrow{F} s'$.

The relation $\cong$ is a congruence of finite index on $\Sigma^*$.
Congruences

Let \([w] \sim\) denote the equivalence class of \(w \in \Sigma^*\) w.r.t. \(\sim\).

**Lemma 1** For any \(w \in \Sigma^*\), \([w] \sim\) is the intersection of all sets of the form \(W_{s,s'}, W^F_{s,s'}, \overline{W}_{s,s'}, \overline{W}^F_{s,s'}\), containing \(w\).

\[
T_w = \bigcap_{w \in W_{s,s'}} W_{s,s'} \cap \bigcap_{w \in W^F_{s,s'}} W^F_{s,s'} \cap \bigcap_{w \in \overline{W}_{s,s'}} \overline{W}_{s,s'} \cap \bigcap_{w \in \overline{W}^F_{s,s'}} \overline{W}^F_{s,s'}
\]

We show that \([w] \sim = T_w\).

“\(\subseteq\)” If \(u \sim w\) then clearly \(u \in T_w\).
Congruences

“⊇” Let \( u \in T_w \)

- if \( s \xrightarrow{w} s' \), then \( w \in W_{s,s'} \), hence \( u \in W_{s,s'} \), then \( s \xrightarrow{u} s' \) as well.

- if \( s \xleftarrow{w} s' \), then \( w \in W_{s,s'} \), hence \( u \in W_{s,s'} \), then \( s \xleftarrow{u} s' \).

Also,

- if \( s \xrightarrow{F_w} s' \), then \( w \in W_{s,s'}^F \), hence \( u \in W_{s,s'}^F \), then \( s \xrightarrow{u} s' \) as well.

- if \( s \xleftarrow{F_w} s' \), then \( w \in W_{s,s'}^F \), hence \( u \in W_{s,s'}^F \), then \( s \xleftarrow{u} s' \).

Then \( u \cong w \).

This lemma gives us a way to compute the \( \cong \)-equivalence classes.
Outline of the proof

We prove that:

$$\mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) \neq \emptyset} VW^\omega$$

where $V, W$ are $\cong$-equivalence classes

Then we have

$$\Sigma^\omega \setminus \mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) = \emptyset} VW^\omega$$

Finally we obtain an algorithm for complementation of Büchi automata
**Saturation**

**Definition 4** A congruence relation $R \subseteq \Sigma^* \times \Sigma^*$ saturates an $\omega$-language $L$ iff for all $R$-equivalence classes $V$ and $W$, if $VW^\omega \cap L \neq \emptyset$ then $VW^\omega \subseteq L$.

**Lemma 2** The congruence relation $\cong$ saturates $\mathcal{L}(A)$. 
Every word belongs to some $VW^\omega$

Let $\alpha \in \Sigma^\omega$ be an infinite word for the rest of this section.

By $\alpha(n, m)$, we denote $\alpha(n)\alpha(n + 1)\ldots\alpha(m - 1)$, $n \leq m$.

We will build two $\cong$-equivalence classes $V$ and $W$ such that $\alpha \in V \cdot W^\omega$.

Together with the saturation lemma, this proves

$$\mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) \neq \emptyset} VW^\omega$$
Merging of positions

Definition 5  Two positions $k, k' \in \mathbb{N}$ are said to merge at $m$, $m > k$ and $m > k'$ iff $\alpha(k, m) \equiv \alpha(k', m)$. We say that $k$ and $k'$ are $\equiv_{\alpha}$-equivalent, denoted $k \equiv_{\alpha} k'$ iff they merge at $m$, for some $m > k, k'$.

If $k$ and $k'$ merge at $m$ then they also merge at $m'$, for all $m' \geq m$.

$k \equiv_{\alpha} k' (m)$ is an equivalence relation on $\mathbb{N}$ of finite index.
Merging of positions

There exists infinitely many positions $0 < k_0 < k_1 < \ldots$, all $\cong_\alpha$-equivalent.

Consider the sequence $\alpha(k_0, k_1), \alpha(k_0, k_2), \alpha(k_0, k_3) \ldots$

There exist $\alpha(k_0, k_{i_0}), \alpha(k_0, k_{i_1}), \alpha(k_0, k_{i_2}) \ldots$ all $\cong$-equivalent

There exist $k_{j_0}, k_{j_1}, k_{j_2}, \ldots$ such that for all $i \leq j$ $k_i \cong_\alpha k_j(k_{j+1})$

There exists infinitely many positions $0 < k_0 < k_1 < k_2 < \ldots$ such that

1. $\alpha(k_0, k_i) \cong \alpha(k_0, k_j)$ for all $i, j \in \mathbb{N}$
2. $k_i \cong_\alpha k_j(k_{j+1})$ for all $i \leq j$. 
**Defining V and W**

Let \( V = [\alpha(0, k_0)] \) and \( W = [\alpha(k_0, k_1)] \).

By (1) \( \alpha(k_0, k_1) \cong \alpha(k_0, k_i) \) for all \( i > 0 \).

By (2) \( \alpha(k_0, k_{i+1}) \cong \alpha(k_i, k_{i+1}) \), for all \( i > 0 \).

By (1) \( \alpha(k_0, k_{i}) \cong \alpha(k_0, k_{i+1}) \) for all \( i > 0 \).

Hence \( \alpha(k_0, k_1) \cong \alpha(k_i, k_{i+1}) \), for all \( i > 0 \).

Therefore \( \alpha \in V \cdot W^\omega \).
Complementation of Büchi Automata

**Theorem 2** For any Büchi automaton $A$ there exists a Büchi automaton $\overline{A}$ such that $\mathcal{L}(\overline{A}) = \Sigma^\omega \setminus \mathcal{L}(A)$.

$$\mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) \neq \emptyset} VW^\omega$$

where $V, W$ are $\cong$-equivalence classes

$$\Sigma^\omega \setminus \mathcal{L}(A) = \bigcup_{VW^\omega \cap \mathcal{L}(A) = \emptyset} VW^\omega$$
An Application of Ramsey Theorem for Infinite Graphs

Theorem 3 (Wikipedia) Let $X$ be some countably infinite set and colour the subsets of $X$ of size $n$ in $c$ different colours. Then there exists some infinite subset $M$ of $X$ such that the size $n$ subsets of $M$ all have the same colour.

Let $X = \langle \mathbb{N}, \{(i, j) \mid i < j\} \rangle$ ($n = 2$). We define the coloring $i \xrightarrow{W} j$ iff $\alpha(i, j) \in W$.

Then there exists an infinite subset $M = \{k_0 < k_1 < \ldots\} \subseteq \mathbb{N}$ and a $\cong$-equivalence class $W$ such that $k_i \xrightarrow{W} k_j$ for all $i < j \in \mathbb{N}$.

We obtain that $\alpha(k_i, k_{i+1})$, for all $i \in \mathbb{N}$. 
Deterministic Büchi Automata

ω-languages recognized by NBA ⊃ ω-languages recognized by DBA

Let $W \subseteq \Sigma^\ast$. Define $\overrightarrow{W} = \{ \alpha \in \Sigma^\omega \mid \alpha(0,n) \in W \text{ for infinitely many } n \}$

Theorem 4 A language $L \subseteq \Sigma^\omega$ is recognizable by a deterministic Büchi automaton iff there exists a rational language $W \subseteq \Sigma^\ast$ such that $L = \overrightarrow{W}$.

If $L = \mathcal{L}(A)$ then $W = \mathcal{L}(A')$ where $A'$ is the DFA with the same definition as $A$, and with the finite acceptance condition.
Deterministic Büchi Automata

**Theorem 5** There exists a Büchi recognizable language that can be recognized by no deterministic Büchi automaton.

\[ \Sigma = \{a, b\} \text{ and } L = \{ \alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty \} = \Sigma^* b^\omega. \]

Suppose \( L = \overrightarrow{W} \) for some \( W \subseteq \Sigma^*. \)

\[ b^\omega \in L \Rightarrow b^{n_1} \in W \]

\[ b^{n_1}ab^\omega \in L \Rightarrow b^{n_1}ab^{n_2} \in W \]

\[ \ldots \]

\[ b^{n_1}ab^{n_2}a\ldots \in \overrightarrow{W} = L, \text{ contradiction.} \]
Deterministic Büchi Automata are not closed under complement

Theorem 6 There exists a DBA $A$ such that no DBA recognizes the language $\Sigma^\infty \setminus L(A)$.

$\Sigma = \{a, b\}$ and $L = \{\alpha \in \Sigma^\omega \mid \#_a(\alpha) < \infty\} = \Sigma^*b^\omega$.

Let $V = \Sigma^*a$. There exists a DFA $A$ such that $L(A) = V$.

There exists a deterministic Büchi automaton $B$ such that $L(A) = \overline{V}$

But $\Sigma^\omega \setminus \overline{V} = L$ which cannot be recognized by any DBA.
**Büchi Automata and S1S**

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Any finite word $w \in \Sigma^*$ induces the *infinite* sets $p_a = \{p \mid w(p) = a\}$.

- $x \leq y$: $x$ is less than $y$,
- $S(x) = y$: $y$ is the successor of $x$,
- $p_a(x)$: $a$ occurs at position $x$ in $w$

Remember that $\leq$ and $S$ can be defined one from another.
Problem Statement

Let $\mathcal{L}(\varphi) = \{ w \mid m_w \models \varphi \}$

A language $L \subseteq \Sigma^*$ is said to be S1S-definable iff there exists a S1S formula $\varphi$ such that $L = \mathcal{L}(\varphi)$.

1. Given a Büchi automaton $A$ build an S1S formula $\varphi_A$ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

2. Given an S1S formula $\varphi$ build a Büchi automaton $A_\varphi$ such that $\mathcal{L}(A) = \mathcal{L}(\varphi)$

The Büchi recognizable and S1S-definable languages coincide
From Automata to Formulae

Let $A = \langle S, I, T, F \rangle$ with $S = \{s_1, \ldots, s_p\}$, and $\Sigma = \{0, 1\}^m$.

Build $\Phi_A(X_1, \ldots, X_m)$ such that $\forall w \in \Sigma^* . w \in \mathcal{L}(A) \iff w \models \Phi_A$

$\Phi_A(X_1, \ldots, X_m) = \exists Y_1 \ldots \exists Y_p . \Phi_S(Y) \land \Phi_I(Y) \land \Phi_T(Y, X) \land \Phi_F(Y)$

$\Phi_F(Y) = \forall x \exists y . x \leq y \land x \neq y \land \bigvee_{s_i \in F} Y_i(y)$
Consequences

Theorem 7  A language $L \subseteq \Sigma^\omega$ is definable in S1S iff it is Büchi recognizable.

Corollary 1  The SAT problem for S1S is decidable.

Lemma 3  Any S1S formula $\phi(X_1, \ldots, X_m)$ is equivalent to an S1S formula of the form $\exists Y_1 \ldots \exists Y_p \cdot \varphi$, where $\varphi$ does not contain other set variables than $X_1, \ldots, X_m, Y_1, \ldots, Y_p$. 
Müller and Rabin Word Automata
Müller Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

Definition 6  A Müller automaton over $\Sigma$ is $A = \langle S, s_0, T, F \rangle$, where:

- $S$ is the finite set of states
- $s_0 \in S$ is the initial state
- $T : S \times \Sigma \mapsto S$ is the transition table
- $F \subseteq 2^S$ is the set of accepting sets

Notice that Müller automata are deterministic and complete by definition.
Acceptance Condition

A run of a Müller automaton is defined over an infinite word \( w : \alpha_1 \alpha_2 \ldots \) as an infinite sequence of states \( \pi : s_0 s_1 s_2 \ldots \) such that:

- \( T(s_i, \alpha_{i+1}) = s_{i+1} \), for all \( i \in \mathbb{N} \).

Let \( \inf(\pi) = \{ s \mid s \text{ appears infinitely often on } \pi \} \).

Run \( \pi \) of \( A \) is said to be accepting iff \( \inf(\pi) \in \mathcal{F} \).

\( L \subseteq \Sigma^\omega \) is Müller-recognizable iff there exists a MA \( A \) such that \( L = \mathcal{L}(A) \).
Theorem 8  For each deterministic Büchi automaton $A$ there exists a Müller automaton $B$ such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $A = \langle S, \{s_0\}, T, F \rangle$ be a deterministic Büchi automaton.

Define $B = \langle S, s_0, T, \{G \in 2^S \mid G \cap F \neq \emptyset\} \rangle$
Closure Properties

**Theorem 9** The class of Müller-recognizable languages is closed under union, intersection and complement.

Let $A = \langle S, s_0, T, \mathcal{F} \rangle$ be a Müller automaton.

Define $B = \langle S, s_0, T, 2^S \setminus \mathcal{F} \rangle$.

We have $\mathcal{L}(B) = \Sigma^\omega \setminus \mathcal{L}(A)$. 
Closure Properties

Let $A_i = \langle S_i, s_{0,i}, T_i, \mathcal{F}_i \rangle$, $i = 1, 2$ be Müller automata.

Define $B = \langle S, s_0, T, \mathcal{F} \rangle$ where:

- $S = S_1 \times S_2$,
- $s_0 = \langle s_{0,1}, s_{0,2} \rangle$,
- $T(\langle s_1, s_2 \rangle, a) = \langle T(s_1, a), T(s_2, a) \rangle$
- $\mathcal{F} = \{\{\langle s_1, s'_1 \rangle, \ldots, \langle s_k, s'_k \rangle\} \mid \{s_1, \ldots, s_k\} \in \mathcal{F}_1 \text{ or } \{s'_1, \ldots, s'_k\} \in \mathcal{F}_2\}$

We have $\mathcal{L}(B) = \mathcal{L}(A_1) \cup \mathcal{L}(A_2)$.

For intersection it is enough to set

$\mathcal{F} = \{\{\langle s_1, s'_1 \rangle, \ldots, \langle s_k, s'_k \rangle\} \mid \{s_1, \ldots, s_k\} \in \mathcal{F}_1 \text{ and } \{s'_1, \ldots, s'_k\} \in \mathcal{F}_2\}$
Characterization of Müller-recognizable languages

A language \( L \subseteq \Sigma^\omega \) is Müller-recognizable iff \( L \) is a Boolean combination of sets \( \overrightarrow{W} \), \( W \subseteq \Sigma^* \), i.e. \( L = \bigcup_i \left( \bigcap_j \overrightarrow{W}_{ij} \cap \bigcap_k (\Sigma^\omega \setminus \overrightarrow{W}_{ik}) \right) \).

“\( \Leftarrow \)” Any set \( \overrightarrow{W}_{ij} \) is recognized by a deterministic Büchi automaton, hence also by a Müller automaton.

“\( \Rightarrow \)” Let \( A = \langle S, s_0, T, \mathcal{F} \rangle \) be a Müller automaton recognizing \( L \).

Let \( A_q = \langle S, s_0, T, \{q\} \rangle \), \( q \in S \), and \( W_q = \mathcal{L}(A_q) \).

\[ L = \bigcup_{Q \in \mathcal{F}} \left( \bigcap_{q \in Q} \overrightarrow{W}_q \cap \bigcap_{q \in S \setminus Q} (\Sigma^\omega \setminus \overrightarrow{W}_q) \right) \]
Exercise

Let $\Sigma = \{a, b\}$ and $A = \langle S, s_0, T, \mathcal{F} \rangle$, where:

- $S = \{s_0, s_1\}$,
- $T(s_0, a) = s_0$, $T(s_0, b) = s_1$, $T(s_1, a) = s_0$ and $T(s_1, b) = s_1$,
- $\mathcal{F} = \{\{s_0, s_1\}\}$

What is $L(A)$? What if $A$ was Büchi with $F = \{s_0, s_1\}$?
Rabin Word Automata

Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

**Definition 7** A Rabin automaton over $\Sigma$ is $A = \langle S, s_0, T, \Omega \rangle$, where:

- $S$ is the finite set of states
- $s_0 \in S$ is the initial state
- $T : S \times \Sigma \mapsto S$ is the transition table
- $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$ is the set of accepting pairs, $N_i, P_i \subseteq S$.

Run $\pi$ of $A$ is said to be *accepting* iff

$$\inf(\pi) \cap N_i = \emptyset \text{ and } \inf(\pi) \cap P_i \neq \emptyset$$

for some $1 \leq i \leq k$. 
The Streett acceptance condition

The Rabin acceptance condition is of the form:

$$\bigvee_{1 \leq i \leq k} \inf(\pi) \cap N_i = \emptyset \land \inf(\pi) \cap P_i \neq \emptyset$$

The Streett acceptance condition is the negation:

$$\bigwedge_{1 \leq i \leq k} \inf(\pi) \cap N_i \neq \emptyset \rightarrow \inf(\pi) \cap P_i \neq \emptyset$$
From Rabin to Müller

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, there exists a Müller automaton $B$ such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\Omega = \{\langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle\}$.

Let $A_i = \langle S, s_0, T, P_i \rangle$, and $B_i = \langle S, s_0, T, N_i \rangle$.

$$\mathcal{L}(A) = \bigcup_{i=1}^{k} \left( \overline{\mathcal{L}(A_i)} \cap (\Sigma^\omega \setminus \overline{\mathcal{L}(B_i)}) \right)$$
From Rabin to Müller (2)

Given a Rabin automaton $A = \langle S, s_0, T, \Omega \rangle$, such that

$$\Omega = \{ \langle N_1, P_1 \rangle, \ldots, \langle N_k, P_k \rangle \}$$

let $B = \langle S, s_0, T, \mathcal{F} \rangle$ be the Müller automaton, where

$$\mathcal{F} = \{ F \subseteq S \mid F \cap N_i = \emptyset \text{ and } F \cap P_i \neq \emptyset \text{ for some } 1 \leq i \leq k \}$$
From Müller to Rabin

Given a Müller automaton $A = \langle S, s_0, T, \mathcal{F} \rangle$, there exists a Rabin automaton $B$ such that $\mathcal{L}(A) = \mathcal{L}(B)$

Let $\mathcal{F} = \{Q_1, \ldots, Q_k\}$

Let $B = \langle S', s'_0, T', \Omega' \rangle$ where:

- $S' = 2^{Q_1} \times \ldots \times 2^{Q_k} \times S$
- $s'_0 = \langle \emptyset, \ldots, \emptyset, s_0 \rangle$
From Müller to Rabin

- $T'(\langle S_1, \ldots, S_k, s \rangle, a) = \langle S'_1, \ldots, S'_k, s' \rangle$ where:
  - $s' = T(s, a)$
  - $S'_i = \emptyset$ if $S_i = Q_i$, $1 \leq i \leq k$
  - $S'_i = (S_i \cup \{s'\}) \cap Q_i$, $1 \leq i \leq k$

- $P_i = \{ \langle S_1, \ldots, S_i, \ldots, S_k, s \rangle \mid S_i = Q_i \}, 1 \leq i \leq k$

- $N_i = \{ \langle S_1, \ldots, S_i, \ldots, S_k, s \rangle \mid s \not\in Q_i \}, 1 \leq i \leq k$
The Big Picture

NBA → MA

DBA

MA → RA

McNaughton

RA
Exercise

Let $A = \langle S, s_0, T, \{Q_1, \ldots, Q_t\} \rangle$ be a Müller automaton. Consider the Rabin automaton $A' = \langle S, s_0, T, \Omega \rangle$ where

$$\Omega = \{(S \setminus Q_1, Q_1), \ldots, (S \setminus Q_t, Q_t)\}$$

Give an example of $A$ such that $\mathcal{L}(A) \neq \mathcal{L}(A')$. 