From simple combinatorial statements with difficult mathematical proofs to hard instances of SAT

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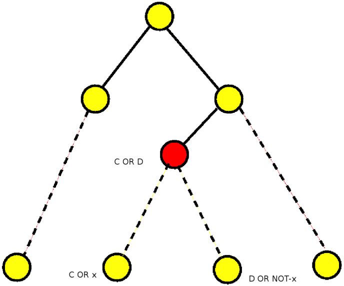
(joint work with Adrian Crãciun)

Caution

- This talk: "Theory A" (proof complexity), unpublished work.
- Naturally continues with experimental work on SAT benchmarks.
- One-line soundbite: Do combinatorial statements with difficult (mathematical) proofs correspond to "hard" instances of SAT ?
- I am not solving any major open problem in computational complexity

Reminder: Propositional proof complexity

- Proving that a formula is not satisfiable seems "harder" than finding a solution.
- Possible: proof systems for unsatisfiability, e.g. resolution
- $C \vee x, D \vee \overline{x} \rightarrow (C \vee D), x, \overline{x} \rightarrow \square$.
- Complexity= minimum length of a resolution proof.
- Lower bound for the running time of all DPLL algorithms !



Reminder: Propositional proof complexity (II)

- Resolution proof size may be exponential
- E.g. Pigeonhole formula(s): PHP_n^{n-1} (Haken)
- $X_{i,j} = 1$ "pigeon i goes to hole j".
- $X_{i,1} \vee X_{i,2} \vee \ldots \vee X_{i,n-1}$, $1 \leq i \leq n$ (each pigeon goes to (at least) one hole)
- $\overline{X_{k,j}} \vee \overline{X_{l,j}}$ (pigeons k and l do not go together to hole j).
- Resolution: clausal formulas. Stronger proof systems ?

Boundaries of proof complexity: Frege proofs

- Example, for concreteness [Hilbert Ackermann]
  - propositional variables $p_1, p_2, \ldots$.
  - Connectives $\neg, \vee$.
  - Axiom schemas:
    1. $\neg(A \vee A) \vee A$
    2. $\neg A \vee (A \vee B)$
    3. $\neg(A \vee B) \vee (B \vee A)$
    4. $\neg(\neg A \vee B) \vee (\neg(C \vee A) \vee (C \vee B))$
  - Rule: From A and $\neg A \vee B$ derive B.
- Cook-Reckhow: all Frege proof systems equivalent (polynomially simulate each other)
- Can prove *PHP* in polynomial size (Buss).
- Still exponential l.b. ($2^{n^{\epsilon}}$) if we restrict formula depth (bounded-depth Frege)

Boundary of knowledge: Frege proofs (II)

- PHP (Buss): proof by counting
- Usual proof by induction: exponential size in Frege: reduction causes formula size to increase by a constant factor at every reduction step.
- Polynomial if we allow introducing new variables: $X \equiv \Phi(\overline{Y})$.
- Frege + new vars: extended Frege

Our original idea/motivation

- Open question: Is extended Frege more powerful than Frege ?
- Most natural candidates for separation turned out to have subexponential Frege proofs.
- Perhaps translating into SAT a mathematical statement that is (mathematically) hard to prove would yield a natural candidate for the separation.
- Didn't quite work out: Our examples probably harder than extended Frege.

Kneser's Conjecture

- Stated in 1955 (Martin Kneser, Jaresbericht DMV)
- Let $n \geq 2k - 1 \geq 1$. Let $c : \binom{n}{k} \to [n - 2k + 1]$. Then there exist two disjoint sets A and B with $c(A) = c(B)$.

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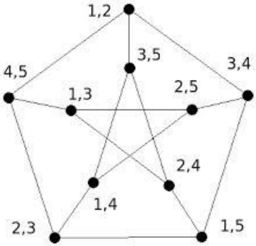
- $k = 1$ Pigeonhole principle !
- $k = 2, 3$ combinatorial proofs (Stahl, Garey & Johnson)
- $k \geq 4$ only proved in 1977 (Lovász) using Algebraic Topology.
- Combinatorial proofs known (Matousek, Ziegler). "hide" Alg. Topology
- No "purely combinatorial" proof known

Kneser's Conjecture (II)

- the chromatic number of a certain graph $Kn_{n,k}$ (at least) $n - 2k + 2$. (exact value)
- Vertices: $\binom{n}{k}$. Edges: disjoint sets.
- E.g. $k = 2, n = 5$: Petersen's graph has chromatic number (at least) three.

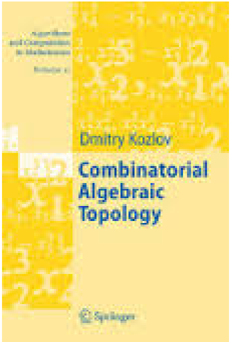


Stronger form: Schrijver's Theorem

- inner cycle in Petersen's graph already chromatic number three.
- $A \in \binom{n}{k}$ stable if it doesn't contain consecutive elements i, $i+1$ (including $n, 1$).
- Schrijver's Theorem: Kneser's conjecture holds when restricted to stable sets only.

Algebraic topology and graph colorings

- Dolnikov's theorem: generalization, lower bounds on the chromatic number of an arbitrary graph.
- In general not tight.
- Many other extensions.



Lovász-Kneser's Thm. as an (unsatisfiable) propositional formula

- naïve encoding $X_{A,k} = TRUE$ iff A colored with color k.
- $X_{A,1} \vee X_{A,2} \vee \ldots \vee X_{A,n-2k+1}$ "every set is colored with (at least) one color"
- $\overline{X_{A,j}} \vee \overline{X_{B,j}}$ $(A \cap B = \emptyset)$ "no two disjoint sets are colored with the same color"
- Fixed k: $Kneser_{k,n}$ has poly-size (in n).
- Extends encoding of PHP

Our results in a nutshell

- $Kneser_{k,n}$ reduces to (is a special case of) $Kneser_{k+1,n-2}$.
- Thus all known lower bounds that hold for PHP (resolution, bd. Frege) hold for any $Kneser_k$.
- Cases with combinatorial proofs:
  - $k = 2$: polynomial size Frege proofs
  - $k = 3$: polynomial size extended Frege proofs
- $k \geq 4$: polynomial size implicit_2 extended Frege proofs
- Implicit proofs: Krajicek (2002). Very powerful proof system(s). AFAIK: first concrete example.

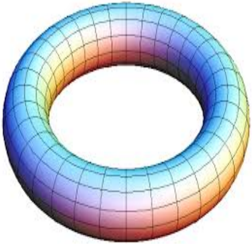
Significance

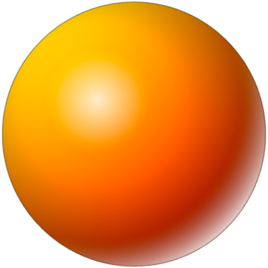
- Proof complexity: counterpart, expressibility in (versions of) bounded arithmetic
- Reverse mathematics: what is the weakest proof system that can prove a certain result ?
- Stephen Cook: "bounded reverse mathematics"

- Implicit proofs seem to be needed for simulating arguments involving algebraic topology.
- Reasons: ~~exponentially large objects~~ and nonconstructive methods
- CONJECTURE: For $k \geq 4$ $Kneser_{k,n}$ requires exponential-size (extended) Frege proofs

What is Algebraic Topology and why can it prove lower bounds on chromatic numbers ?

- Two objects similar if can continuously morph one into the other
- Cannot turn a donut into a sphere: Hole is an "obstruction" to contracting a circle going around the torus to a point.
- Can do that on a sphere.
- Continuous morphing should preserve contractibility.



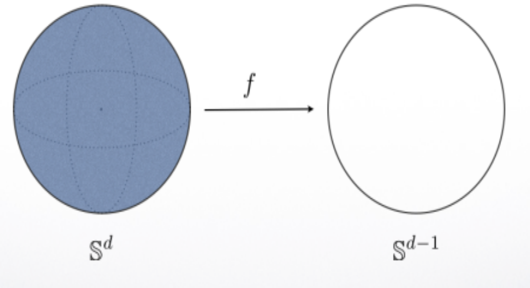


How do we "measure" the "number of holes" (and other properties) ?

- algebraic objects (groups)
- Functorial: $G \to H$ implies $F(G) \to F(H)$.
- If $K \to F(G)$ but $K \not\to F(H)$ then K acts as an obstruction to $G \to H$
- Coloring = morphism of graphs.

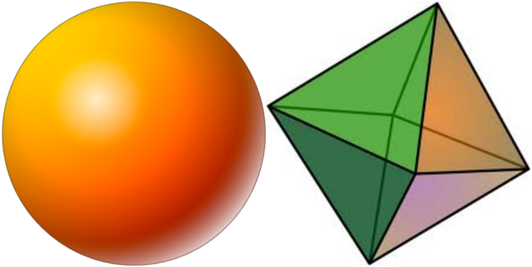
Ingredient of Kneser proof: Borsuk-Ulam Thm.

- Cannot map continuously and antipodally n-dim. sphere into a sphere of lower dimension (or ball into sphere)
- Obstruction: largest dimension of sphere that can be embedded continuously and antipodally into $F(G)$. As long as $F(K_m)$ "is a sphere".



From continuous to discrete

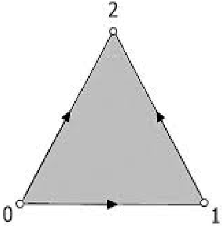
- A sphere is topologically equivalent to an octahedron
- simplicial complex: every subset of a face is a face.
- Simplex: purely combinatorially (sets that are simplices)

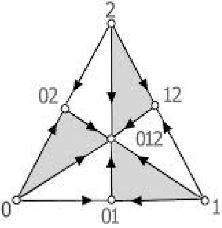


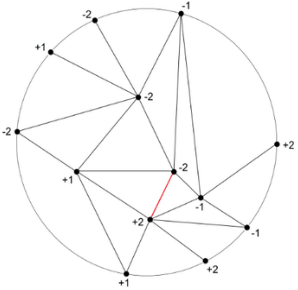
- Vertices: $\{\pm 1, \pm 2, \ldots, \pm n\}$.
- Faces: subsets that do not contain no i and $-i$.
- Exponentially (in n) many faces !

Discrete Borsuk-Ulam: Tucker's lemma

- Antipodally Symmetric Triangulation T of the n-ball. Barycentric subdivision, one vertex for each face
- For any labeling of T with vertices from $\{\pm 1, \ldots, \pm(n-1)\}$ antipodal on the boundary there exist two adjacent vertices $v \sim w$ with $c(v) = -c(w)$.
- Intuition: no continuous (a.k.a simplicial) antipodal map from the n-ball to the n-sphere.







KNESER FROM TUCKER ($k \geq 4$)

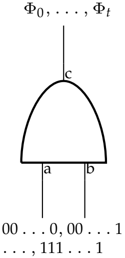
- Simulate "combinatorial" proof of Kneser (combination of two mathematical proofs)
- Tucker's lemma: unsatisfiable propositional formula. $Kneser_{k,n}$: variable substitution.
- barycentric dimension ⇒ exponentially large formula !
- Kneser follows from a new "low dimensional" Tucker lemma.
- Avoid barycentric subdivision. Instead (k+k) "skeleton"

Kneser from Tucker ($k \geq 4$)

- Second obstacle: Tucker lemma is nonconstructive (PPAD complete).
- Given an (exponential size) graph with one vertex of odd degree, find another node of odd degree
- For Kneser: this exponential graph has very regular structure.

Implicit proofs

- Krajicek (J. Symb. Logic 2004).
- Hierarchy: $iEF, i_2EF, i_3EF, \ldots$.
- ridiculously powerful: implicit resolution $\equiv$ extended Frege.
- poly-size boolean circuit that is generating all formulas in an extended Frege proof + correctness proof
- if correctness proof itself implicit $\Rightarrow$ second level. Correctness proof second level $\Rightarrow$ third level . . .



Implicit proofs: Kneser

- polynomial number of output gates $\Rightarrow \Phi_0, \ldots, \Phi_t$ "small"
- extended Frege: renaming keeps formulas small.
- implicit proofs allows us to generate a proof of the odd degree argument
- soundness: exponentially large (but regular) $\Rightarrow$ Kneser: second level

REDUCING $Kneser_{n,k+1}$ TO $Kneser_{n-2,k}$

- There exists a variable substitution $\Phi_k : Var(Kneser_{n,k+1}) \to Var(Kneser_{n-2,k})$ s.t. $\Phi_k(Kneser_{n,k+1})$ consists precisely of the clauses of $Kneser_{n-2,k}$ (perhaps repeated and in a different order)
- Let $A \in \binom{n}{k+1}$. Define $\Phi_k(X_{A,i})$ by:
  - **Case 1:** $A_{\leq k} \subseteq [n-2]$**:** $\Phi_k(X_{A,i}) = Y_{A_{\leq k},i}$
  - **Case 2:** $A_{\leq k} \not\subseteq [n-2]$**:** $(n-1, n \in A)$
    Let $A = P \cup \{n-1, n\}$, $|P| = k-1$. Let $\lambda = max\{j : j \leq n-2, j \notin P\}$. Define $\Phi_k(X_{A,i}) = Y_{P\cup\{\lambda\},i}$
- Clause $X_{A,1} \vee X_{A,2} \vee \ldots \vee X_{A,n-2k+1}$ maps to $Y_{B,1} \vee Y_{B,2} \vee \ldots \vee Y_{B,n-2k+1}$, $B = A$ (Case 1).
- Clauses $\overline{X_{A,i}} \vee \overline{X_{B,i}}$ $(A \cap B = \emptyset)$ map to $\overline{Y_{C,i}} \vee \overline{Y_{D,i}}$
- Case 2 cannot happen for both A and B. By case analysis $C \cap D = \emptyset$.

Comments on (other) proofs

- Lower bounds Schrijver: Same substitution, slightly more complicated argument.
- $k = 2$: counting proof, Stahl+ Buss PHP.
- For any color class $c^{-1}(\lambda)$ one of the following is true (assuming conclusion of Kneser does not hold):
  - $|c^{-1}(\lambda)| \leq 3$.
  - All sets $B \in c^{-1}(\lambda)$, $|c^{-1}(\lambda)| \geq 4$, have one element in common (call such an element special).
  - Frege systems can "count" (employing techniques developed by Buss) the number of special elements.
- $k = 3$: Counting approach fails (technical reasons), have to settle for extended Frege.

From Kneser-like results to hard SAT instances ?

- $2^{\Omega(n)}$ resolution complexity. Are they hard in practice ?
- At this point: only idea for subsequent work
- Want: small formulas.
- $Kneser_{n,k}$: $\sim n^{k+1}$ variables, even more clauses.
- Schrijver ? Other versions of Dolnikov's Theorem ? expander graph with tight bounds on the chromatic number
- Better encodings ? All intuitions should apply.
- Kneser, stable Kneser graphs: symmetries well understood. But: reason for unsatisfiability is more global

Further possible work

- Other proof systems: e.g. cutting planes (k=2), polynomial calculus, etc.
- (in progress) Topological obstructions: from graph coloring to CSP.
- Logics for implicit proof systems ?
- Topological arguments as sound (but incomplete) implicit proof systems
  - if $K \not\to L$ then a "proof of $A \not\to B$" is a pair of embeddings $(K \to A), (B \to L)$.
  - Checking soundness ($K \not\to L$) may not be polynomial. If K, L "standard objects" we could omit proof of $K \not\to L$ from complexity
- Automated theorem proving ?

Thank you. Questions ?