Homework

Problem 1

A *prefix rewriting system* (PRS) is a pair $P = \langle \Gamma, T \rangle$, where:

- Γ is a finite alphabet,
- T is a set of transition rules of the form:

$$\gamma \hookrightarrow w$$

  where $\gamma \in \Gamma$ and $w \in \Gamma^*$ is a (possibly empty) word.

If $\gamma \hookrightarrow w$ is a transition rule of P, then for all $v \in \Gamma^*$, we say that γv is an *immediate predecessor* of wv, denoted by $\gamma v \Rightarrow wv$. If $L \subseteq \Gamma^*$ is a language, let:

$$\begin{array}{rcl} pre(L) & = & \{u \mid \exists v \in L \,.\, u \Rightarrow v\} \\ pre^*(L) & = & \{u \mid \exists v \in L \,.\, u \Rightarrow^* v\} \end{array}$$

where $\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$. Suppose that $L \subseteq \Gamma^*$ is a rational language. Show that:

4.1) $pre(L)$ is rational,

4.2) $pre^*(L)$ is rational.

Problem 2

Let Σ be an alphabet. A *context-free language* over Σ is a language of the form:

$$L = \{w \text{ is the frontier of } t \text{ read from left to right} \mid t \in T\}$$

for some rational tree language T over Σ.

3.1) Prove that $\{a^n b^n \mid n \in \mathbb{N}\}$ is a context-free language.

3.2) Prove that $\{a^n b^n c^n \mid n \in \mathbb{N}\}$ is not a context-free language.

Problem 3

An *integer linear set* is a subset of Z^n, $n > 0$, of the form $\{\sum_{i=0}^{k} \lambda_i a_i + b \mid \lambda_1, \ldots, \lambda_k \in \mathbb{Z}\}$ for some $a_1, \ldots, a_k, b \in \mathbb{Z}^n$. An integer semilinear set is a finite union of integer linear sets.

Prove or disprove that:

- Presburger Arithmetic (PA) interpreted over the structure $(\mathbb{Z}, 0, s, +)$ defines precisely integer semilinear sets, i.e. each integer semilinear set $M \in \mathbb{Z}^n$ corresponds to a formula $\phi(x_1, \ldots, x_n)$ in the sense $\langle z_1, \ldots, z_n \rangle \in M \iff \phi(z_1, \ldots, z_n)$ holds, and viceversa.
- the set of positive integers is not definable in PA over the structure $(\mathbb{Z}, 0, s, +)$

Problem 4

Prove that the following sets are not definable in PA interpreted over $(\mathbb{N}, 0, s, +)$:

- the set of primes
- the set of powers of 2