

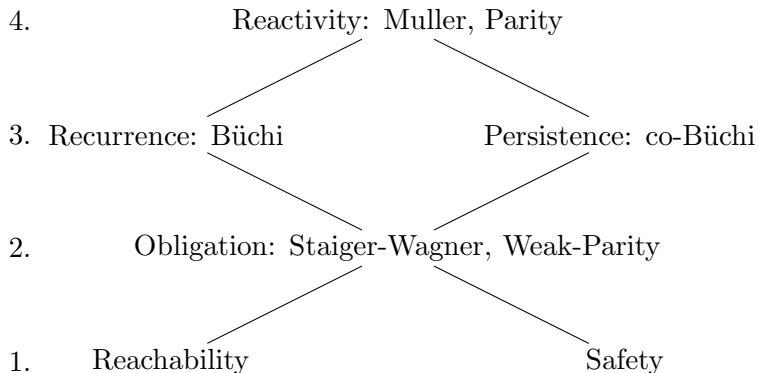
# Reachability and Büchi Games

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# Hierarchy



# Reachability and Safety Games

# Reachability and Safety Games

## Theorem

*Given a reachability game  $(G, F)$  with  $G = (S, S_0, E)$  and  $F \subseteq S$ , then the winning regions  $W_0$  and  $W_1$  of Player 0 and 1, respectively, are computable, and both players have corresponding memoryless winning strategies.*

## Proof.

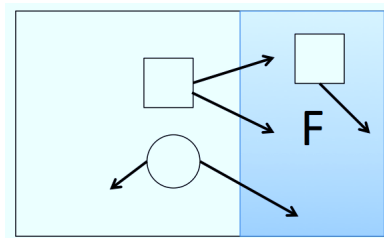
Define

$$\text{Attr}_0^i(F) := \{s \in S \mid \text{Player 0 can force a visit from } s \text{ to } F \\ \text{in less than } i \text{ moves}\}$$

## Force Visit in Next Step

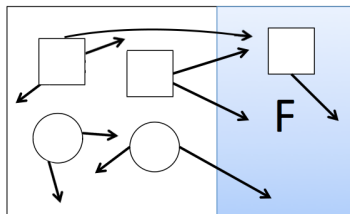
Given a set of states, compute the set of states  $\text{ForceNext}_0(F)$  from which of Player 0 can force to visit  $F$  in the next step. I.e., for each state  $s \in \text{ForceNext}_0(F)$  Player 0 can fix a strategy s.t. all plays starting in  $s$  visit  $F$  in the first step.

$$\begin{aligned} \text{ForceNext}_0(F) = & \{s \in S_0 \mid \exists s' \in S : (s, s') \in E \wedge s' \in \text{Attr}_0^i(F)\} \cup \\ & \{s \in S_1 \mid \forall s' \in S : (s, s') \in E \rightarrow s' \in \text{Attr}_0^i(F)\} \end{aligned}$$



## Observations about ForceNext( $F$ )

1. Player 0 has a **memoryless winning strategy** from ForceNext( $F$ )
2. Player 1 has a **memoryless winning strategy** from  $S \setminus \text{ForceNext}(F)$
3.  $S \setminus \text{ForceNext}(F)$  is a **sub-game**, i.e., every state has at least one successor.
4.  $S \setminus \text{ForceNext}(F)$  is a **trap** for Player 0, i.e., she cannot leave this set.



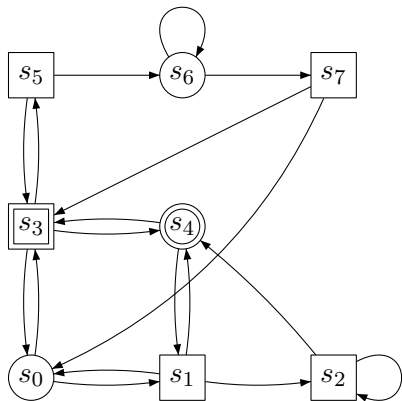
# Computing the Attractor

Construction of  $\text{Attr}_0^i(F)$ :

$$\text{Attr}_0^0(F) = F$$

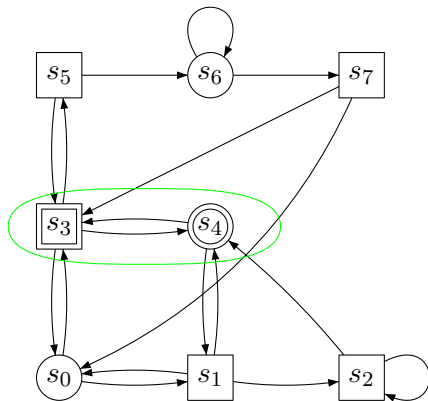
$$\text{Attr}_0^{i+1}(F) = \text{Attr}_0^i(F) \cup \text{ForceNext}_0(\text{Attr}_0^i(F))$$

## Example



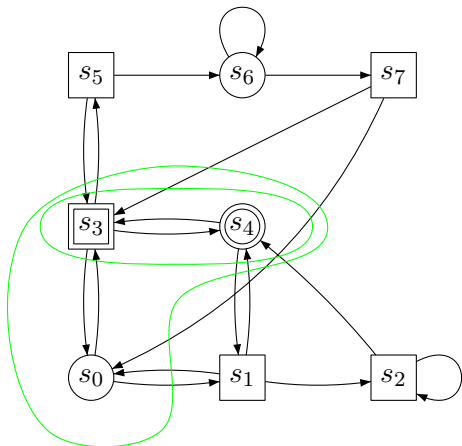


## Example



$$\text{Attr}_0^0 = \{s_3, s_4\}$$

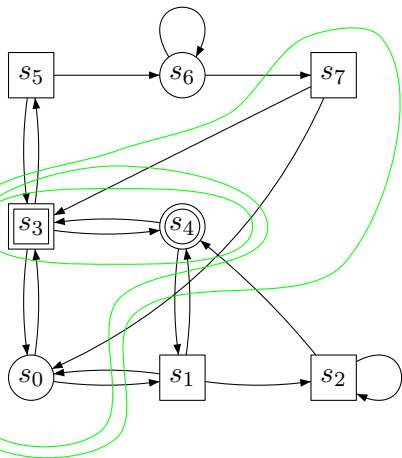
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$$\text{Attr}_0^0 = \{s_3, s_4\}$$

$$\text{Attr}_0^1 = \{s_0, s_3, s_4\}$$

## Example

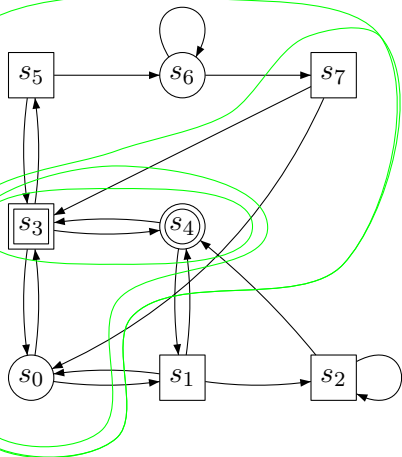


$$\text{Attr}_0^0 = \{s_3, s_4\}$$

$$\text{Attr}_0^1 = \{s_0, s_3, s_4\}$$

$$\text{Attr}_0^2 = \{s_0, s_3, s_4, s_7\}$$

## Example



$$\text{Attr}_0^0 = \{s_3, s_4\}$$

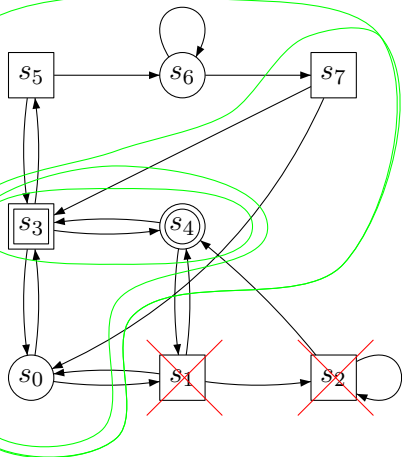
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## Example



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## Computing the Attractor

Construction of  $\text{Attr}_0^i(F)$ :

$$\begin{aligned}\text{Attr}_0^0(F) &= F \\ \text{Attr}_0^{i+1}(F) &= \text{Attr}_0^i(F) \cup \text{ForceNext}_0(\text{Attr}_0^i(F))\end{aligned}$$

Then  $\text{Attr}_0^0(F) \subseteq \text{Attr}_0^1(F) \subseteq \text{Attr}_0^2(F) \subseteq \dots$  and since  $S$  is finite, there exists  $k \leq |S|$  s.t.  $\text{Attr}_0^k(F) = \text{Attr}_0^{k+1}(F)$ .

The 0-Attractor is defined as:

$$\text{Attr}_0(F) := \bigcup_{i=0}^{|S|} \text{Attr}_0^i(F)$$

## Computing the Attractor

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Then  $\text{Attr}_0^0(F) \subseteq \text{Attr}_0^1(F) \subseteq \text{Attr}_0^2(F) \subseteq \dots$  and since  $S$  is finite, there exists  $k \leq |S|$  s.t.  $\text{Attr}_0^k(F) = \text{Attr}_0^{k+1}(F)$ .

The 0-Attractor is defined as:

$$\text{Attr}_0(F) := \bigcup_{i=0}^{|S|} \text{Attr}_0^i(F)$$

Claim:  $W_0 = \text{Attr}_0(F)$  and  $W_1 = S \setminus \text{Attr}_0(F)$

## Duality Between Players

Assume we have a partition of the state space  $S = P_0 \cup P_1$  (i.e.,  $P_0 \cap P_1 = \emptyset$ ) and we want to prove  $W_0 = P_0$  and  $W_1 = P_1$ .

We want to prove  $P_0 \supseteq W_0$ ,  $P_0 \subseteq W_0$ ,  $P_1 \supseteq W_1$ , and  $P_1 \subseteq W_1$ .

Since we know that  $W_0 \cap W_1 = \emptyset$  holds, it is sufficient to prove  $P_0 \subseteq W_0$  and  $P_1 \subseteq W_1$ .

$$\begin{array}{ll} P_0 \subseteq W_0 & P_1 \subseteq W_1 \\ S \setminus P_0 \supseteq S \setminus W_0 & S \setminus P_1 \supseteq S \setminus W_1 \\ P_1 \supseteq S \setminus W_0 \supseteq W_1 & P_0 \supseteq S \setminus W_1 \supseteq W_0 \\ P_1 \supseteq W_1 & P_0 \supseteq W_0 \end{array}$$



## 0-Attractor

To show  $W_0 = \text{Attr}_0(F)$  and  $W_1 = S \setminus \text{Attr}_0(F)$ , we construct winning strategies for Player 0 and 1.

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Proof.

$$\text{Attr}_0(F) \subseteq W_0$$

We prove for every  $i$  and for every state  $s \in \text{Attr}_0^i(F)$  that Player 0 has a positional winning strategy to reach  $F$  in  $\leq i$  steps.

- ▶ (Base)  $s \in \text{Attr}_0^0(F) = F$
- ▶ (Induction)  $s \in \text{Attr}_0^{i+1}(F)$

If  $s \in \text{Attr}_0^i(F)$ , then we apply induction hypothesis.

Otherwise  $s \in \text{ForceNext}_0(\text{Attr}_0^i(F)) \setminus \text{Attr}_0^i(F)$  and Player 0 can force a visit to  $\text{Attr}_0^i(F)$  in one step and from there she needs at move  $i$  steps by induction hypothesis. So,  $F$  is reached after a finite number of moves.

## 0-Attractor cont.

Proof cont.

$$S \setminus \text{Attr}_0(F) \subseteq W_1$$

Assume  $s \in S \setminus \text{Attr}_0(F)$ , then  $s \notin \text{ForceNext}_0(\text{Attr}_0(F))$  and we have two cases:

$$(a) \quad s \in S_0 \cap S \setminus \text{Attr}_0(F) \quad \forall s' \in S: (s, s') \in E \rightarrow s' \notin \text{Attr}_0(F)$$

$$(b) \quad s \in S_1 \cap S \setminus \text{Attr}_0(F) \quad \exists s' \in S: (s, s') \in E \wedge s' \notin \text{Attr}_0(F)$$

In  $S \setminus \text{Attr}_0(F)$  Player 1 can choose edges according to (b) leading again to  $S \setminus \text{Attr}_0(F)$  and by (a) Player 0 cannot escape from  $S \setminus \text{Attr}_0(F)$ . So,  $F$  will be avoided forever.

$$W_0 = \text{Attr}_0(F) \text{ and } W_1 = S \setminus \text{Attr}_0(F)$$

## Safety Games

Given a safety game  $(G, F)$  with  $G = (S, S_0, E)$ , i.e.,

$$\phi_S = \{\rho \in S^\omega \mid \text{Occ}(\rho) \subseteq F\},$$

consider the reachability game  $(G, S \setminus F)$ , i.e.,

$$\phi_R = \{\rho \in S^\omega \mid \text{Occ}(\rho) \cap (S \setminus F) \neq \emptyset\}.$$

$$\begin{aligned} \text{Then, } S^\omega \setminus \phi_R &= \{\rho \in S^\omega \mid \text{Occ}(\rho) \cap (S \setminus F) = \emptyset\} \\ &= \{\rho \in S^\omega \mid \text{Occ}(\rho) \subseteq F\}. \end{aligned}$$

Player 0 has a safety objective in  $(G, F)$ .

Player 1 has a reachability objective in  $(G, F)$ .

So,  $W_0$  in the safety game  $(G, F)$  corresponds to  $W_1$  in the reachability game  $(G, S \setminus F)$ .

## Summary

We know how to solve reachability and safety games by positional winning strategies.

The strategies are

- ▶ Player 0: Decrease distance to  $F$
- ▶ Player 1: Stay outside of  $\text{Attr}_0(F)$

In LTL,  $\diamond F$  = reachability and  $\square F$  = safety.

Next,  $\square\diamond F$  = Büchi and  $\diamond\square F$  = Co-Büchi.

## Exercise

1. Given a reachability game  $(G, F)$  with  $G = (S, S_0, E)$  and  $F \subseteq Q$ , give an algorithm that computes the 0-Attractor( $F$ ) in time  $O(|E|)$ .

## Exercise

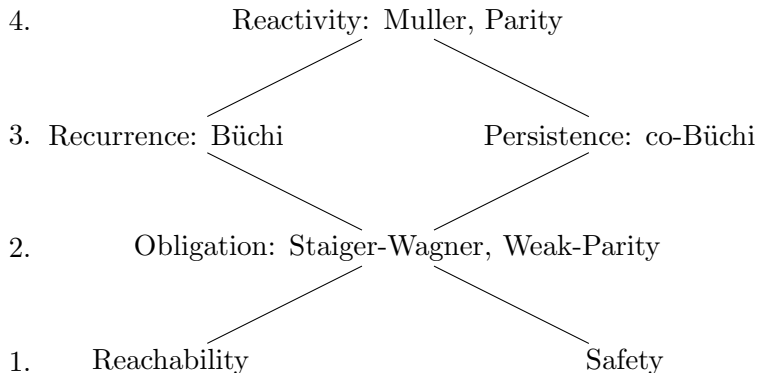
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Solution:

1. Preprocessing: Compute for every state  $s \in S_1$  outdegree  $\text{out}(s)$
2. Set  $n(s) := \text{out}(s)$  for each  $s \in S_1$
3. To breadth-first search backwards from  $F$  with the following conventions:
  - ▶ mark all  $s \in F$
  - ▶ mark  $s \in S_0$  if reached from marked state
  - ▶ mark  $s \in S_1$  if  $n(s) = 0$ , other set  $n(s) := n(s) - 1$ .

The marked vertices are the ones of  $\text{Attr}_0(F)$ .

# Hierarchy





## Büchi and co-Büchi Games

## Büchi Game

Given a Büchi game  $(G, F)$  over the game graph  $G = (S, S_0, E)$  with the set  $F \subseteq S$  of **Büchi states**, we aim to

- ▶ determine the winning regions of Player 0 and 1
- ▶ compute their respective winning strategies

Recall, Player 0 wins  $\rho$  iff she visits infinitely often states in  $F$ , i.e.,  
 $\phi = \{\rho \in S^\omega \mid \text{inf}(\rho) \cap F \neq \emptyset\}$ .

## Idea

Compute for  $i \geq 1$  the set  $\text{Recur}_0^i$  of **accepting** states  $s \in F$  from which Player 0 can force at least  $i$  revisits to  $F$ .

Then, we will show that

$$F \supseteq \text{Recur}_0^1(F) \supseteq \text{Recur}_0^2(F) \supseteq \dots$$

and we compute the winning region of Player 0 with

$$\text{Recur}_0(F) := \bigcap_{i \leq 1} \text{Recur}_0^i(F)$$

Again, since  $F$  is finite, there exists  $k$  such that

$$\text{Recur}_0(F) = \text{Recur}_0^k(F).$$

**Claim:**  $W_0 = \text{Attr}_0(\text{Recur}_0(F))$

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**Claim:**  $W_0 = \text{Attr}_0(\text{Recur}_0(F))$

First, we define  $\text{Recur}_0$  formally using a modified version of Attractor.

## One-Step Attractor

We count **revisits**, so we need the set of states from which Player 0 can force a revisit to  $F$ , i.e., state from which she can force a visit in  $\geq 1$  steps.

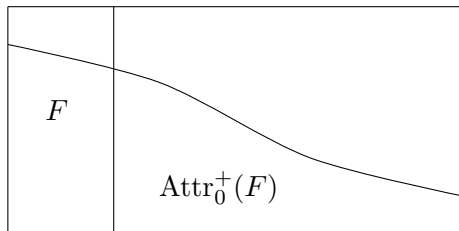
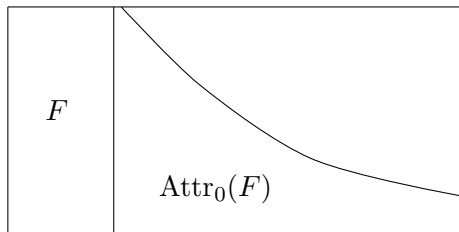
We define a slightly modified attractor:

$$\begin{aligned}A_0^0 &= \emptyset \\A_0^{i+1} &= A_0^i \cup \text{ForceNext}(A_0^i \cup F)\end{aligned}$$

$$\text{Attr}_0^+(F) = \bigcup_{i \geq 0} A_0^i$$

$\text{Attr}_0^+(F)$  is the set of states from which Player 0 can force a revisit to  $F$ .

## Visit versus Revisit



## Recurrence Set

We define

$$\text{Recur}_0^0(F) := F$$

$$\text{Recur}_0^{i+1}(F) := F \cap \text{Attr}_0^+(\text{Recur}_0^i(F))$$

$$\text{Recur}_0(F) := \bigcap_{i \geq 0} \text{Recur}_0^i(F)$$

We show that there exists  $k$  such that  $\text{Recur}_0(F) := \bigcap_{i \geq 0}^k \text{Recur}_0^i(F)$  by proving  $\text{Recur}_0^{i+1}(F) \subseteq \text{Recur}_0^i(F)$  for all  $i \geq 0$ .

**Proof.**

►  $i = 0$ :  $F \cap \text{Attr}_0^+(F) \subseteq F$

►  $i \rightarrow i + 1$ :

$$\begin{aligned} \text{Recur}_0^{i+1}(F) &= F \cap \text{Attr}_0^+(\text{Recur}_0^i(F)) \subseteq F \cap \text{Attr}_0^+(\text{Recur}_0^{i-1}(F)) \\ &= \text{Recur}_0^i(F) \text{ since (i) } \text{Recur}_0^i(F) \subseteq \text{Recur}_0^{i-1}(F) \text{ by ind. hyp. and} \\ &\text{(ii) } \text{Attr}_0^+ \text{ is monotone.} \end{aligned}$$

## Recurrence Set cont.

We show that all states in  $\text{Attr}_0(\text{Recur}_0(F))$  are winning for Player 0, i.e.,  $\text{Attr}_0(\text{Recur}_0(F)) \subseteq W_0$ . We construct a memoryless winning strategy for Player 0 for all states in  $\text{Attr}_0(\text{Recur}_0(F))$ .

Proof.

We know that there exists  $k$  such that

$\text{Recur}_0^{k+1}(F) = \text{Recur}_0^k(F) = F \cap \text{Attr}_0^+(\text{Recur}_0^k(F))$ . So,

- ▶ for  $s \in \text{Recur}_0^k(F) \cap S_0$  Player 0 can choose an edges back to  $\text{Attr}_0^+(\text{Recur}_0^k(F))$  and
- ▶ for  $s \in \text{Recur}_0^k(F) \cap S_1$  all edges lead back to  $\text{Attr}_0^+(\text{Recur}_0^k(F))$ .

For all states in  $\text{Attr}_0(\text{Recur}_0(F)) \setminus \text{Recur}_0(F)$ , Player 0 can follow the attractor strategy to reach  $\text{Recur}_0(F)$ .





## Recurrence Set cont.

We show  $S \setminus \text{Attr}_0(\text{Recur}_0(F)) \subseteq W_1$ .

Proof.

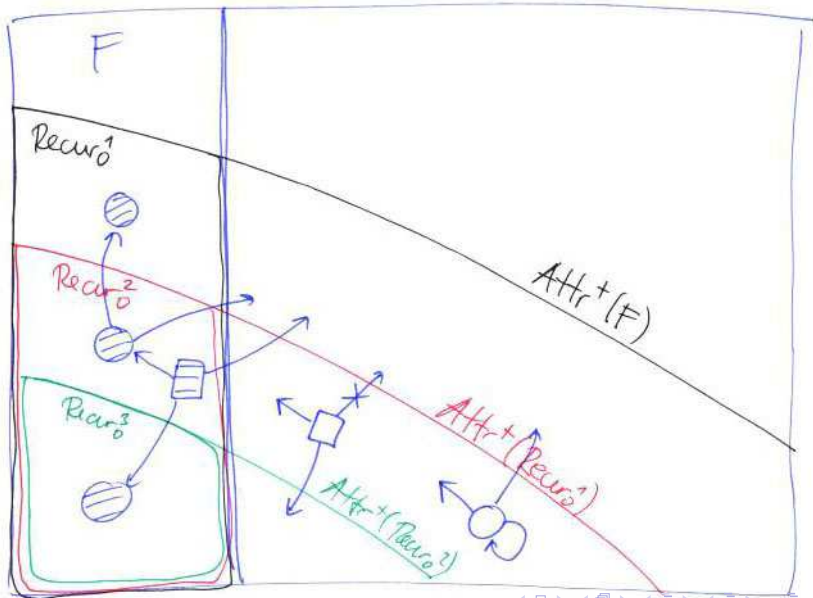
Show: Player 1 can force  $\leq i$  visits to  $F$  from  $s \notin \text{Attr}_0(\text{Recur}_0^i(F))$

$i = 0$ :  $s \notin \text{Attr}_0(F)$ , so Player 1 can avoid visiting  $F$  at all.

$i \rightarrow i + 1$ :  $s \notin \text{Attr}_0(\text{Recur}_0^{i+1}(F))$ .

- ▶  $s \notin \text{Attr}_0(\text{Recur}_0^i(F))$ , Player 1 plays according to ind. hypothese
- ▶ Otherwise,  $s \in \text{Attr}_0(\text{Recur}_0^i(F)) \setminus \text{Attr}_0(\text{Recur}_0^{i+1}(F))$  and Player 1 can avoid  $\text{Attr}_0(\text{Recur}_0^{i+1}(F))$ . In particular,  $s \notin \text{Recur}_0^{i+1}(F) = F \cap \text{Attr}_0^+(\text{Recur}_0^i(F))$ .
  - ▶ If  $s \in \text{Recur}_0^i$ , then Player 1 can force to leave  $\text{Attr}_0^+(\text{Recur}_0^i(F))$ , otherwise  $s \in \text{Recur}_0^{i+1}(F)$ . (So, by ind. hyp. at most  $i + 1$  visits.)
  - ▶ If  $s \in \text{Attr}_0(\text{Recur}_0^i(F)) \setminus \text{Recur}_0^i(F)$ , avoid  $\text{Attr}_0(\text{Recur}_0^{i+1}(F))$ .

# Recurrence Set cont.



## Büchi games

We have shown that Player 0 has a (memoryless) winning strategy from every state in  $\text{Attr}_0(\text{Recur}_0(F))$ , so  $\text{Attr}_0(\text{Recur}_0(F)) \subseteq W_0$ . And, Player 1 has a (memoryless) winning strategy from every state in  $S \setminus \text{Attr}_0(\text{Recur}_0(F))$ , so  $S \setminus \text{Attr}_0(\text{Recur}_0(F)) \subseteq W_1$ . This implies the following theorem.

### Theorem

*Given a Büchi game  $((S, S_0, E), F)$ , the winning regions  $W_0$  and  $W_1$  are computable and form a partition, i.e.,  $W_0 \cup W_1 = S$ . Both players have memoryless winning strategies.*

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### Theorem

*Given a Büchi game  $((S, S_0, E), F)$ , the winning regions  $W_0$  and  $W_1$  are computable and form a partition, i.e.,  $W_0 \cup W_1 = S$ . Both players have memoryless winning strategies.*

How expensive is the computation of  $W_0$  and  $W_1$ ?

## Co-Büchi Games

Given a Co-Büchi Game  $((S, S_0, E), F)$ , i.e.,

$$\phi_C = \{\rho \in S^\omega \mid \text{Inf}(\rho) \subseteq F\}$$

consider the Büchi Game  $((S, S_0, E), S \setminus F)$ , i.e,

$$\phi_B = \{\rho \in S^\omega \mid \text{Inf}(\rho) \cap S \setminus F \neq \emptyset\}.$$

$$\begin{aligned} \text{Then, } S^\omega \setminus \phi_B &= \{\rho \in S^\omega \mid \text{Inf}(\rho) \cap (S \setminus F) = \emptyset\} \\ &= \{\rho \in S^\omega \mid \text{Inf}(\rho) \subseteq F\}. \end{aligned}$$

Player 0 has a co-Büchi objective in  $(G, F)$ .

Player 1 has a Büchi objective in  $(G, F)$ .

So,  $W_0$  in the co-Büchi game  $(G, F)$  corresponds to  $W_1$  in the Büchi game  $(G, S \setminus F)$ .

## Summary

We know how to solve Büchi and Co-Büchi games by positional winning strategies.

In LTL,

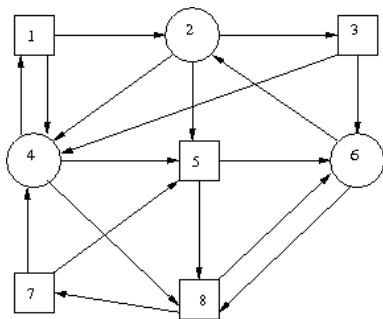
- ▶  $\diamond F$  = reachability
- ▶  $\square F$  = safety
- ▶  $\square \diamond F$  = Büchi
- ▶  $\diamond \square F$  = Co-Büchi

## Exercise

2. Consider the game graph shown in below and the following winning conditions:

- (a)  $\text{Occ}(\rho) \cap \{1\} \neq \emptyset$  and
- (b)  $\text{Occ}(\rho) \subseteq \{1, 2, 3, 4, 5, 6\}$  and
- (c)  $\text{Inf}(\rho) \cap \{4, 5\} \neq \emptyset$ .

Compute the winning regions and corresponding winning strategies showing the intermediate steps (i.e., the Attractor and Recurrence sets) of the computation.



## Exercise

3. Given a game graph  $G = (S, S_0, T)$  and a set  $F \subseteq S$ . Let  $W_0$  and  $W_1$  be the winning regions of Player 0 and Player 1, respectively, in the Buchi game  $(G, F)$ . Prove or disprove:
- (a) The winning region of Player 0 in the safety game  $(G, W_0)$  is  $W_0$ ,
  - (b) If  $f_0$  is a winning strategy for Player 0 in the safety game for  $(G, W_0)$ , then  $f_0$  is also a winning strategy for Player 0 in the Buchi game for  $(G, F)$ ,
  - (c) the winning region of Player 1 in the guaranty game for  $(G, W_0)$  is  $W_1$ , and
  - (d) if  $f_1$  is a winning strategy for Player 1 in the guaranty game for  $(G, W_0)$ , then  $f_1$  is also a winning strategy for Player 1 in the Buchi game  $(G, F)$ .