# Regular, Star Free and Aperiodic Languages

# Regular Languages

Let $\Sigma$ be an alphabet, and $X, Y \subseteq \Sigma^*$

$$
\begin{aligned}
XY &= \{xy \mid x \in X \text{ and } y \in Y\} \\
X^* &= \{x_1 \dots x_n \mid n \geq 0,\ x_1, \dots, x_n \in X\}
\end{aligned}
$$

The class of *regular languages* $\mathcal{R}(\Sigma)$ is the smallest class of languages $L \subseteq \Sigma^*$ such that:

- $\emptyset \in \mathcal{R}(\Sigma)$
- $\{\alpha\} \in \mathcal{R}(\Sigma)$, for all $\alpha \in \Sigma$
- if $X, Y \in \mathcal{R}(\Sigma)$ then $X \cup Y, XY, X^* \in \mathcal{R}(\Sigma)$

# Regular, rational and recognizable languages

**Theorem 1 (Kleene)** *A set of finite words is recognizable if and only if it is regular.*

Proof in every textbook.

Rational = regular, in older books e.g.

Samuel Eilenberg. *Automata, Languages and Machines*. Academic Press, 1974

## $\omega$-Regular Languages

If $X \subseteq \Sigma^*$ and $Y \subseteq \Sigma^\omega$

$$\begin{aligned} XY &= \{xy \mid x \in X,\ y \in Y\} \in \Sigma^\omega \\ X^\omega &= \{x_0 x_1 \ldots \mid x_0, x_1, \ldots \in X \setminus \{\epsilon\}\} \\ X^\infty &= X^* \cup X^\omega \end{aligned}$$

The class of *$\omega$-regular languages* $\mathcal{R}^\infty(\Sigma)$ is the smallest class of languages $L \subseteq \Sigma^\infty$ such that:

- $\emptyset \in \mathcal{R}^\infty(\Sigma)$ and $\{a\} \in \mathcal{R}^\infty(\Sigma)$, for all $a \in \Sigma$
- if $X, Y \in \mathcal{R}^\infty(\Sigma)$ then $X \cup Y \in \mathcal{R}^\infty(\Sigma)$
- for each $X \subseteq \Sigma^*$ and $Y \subseteq \Sigma^\infty$, if $X, Y \in \mathcal{R}^\infty(\Sigma)$ then $XY \in \mathcal{R}^\infty(\Sigma)$
- for each $X \subseteq \Sigma^*$, if $X \in \mathcal{R}^\infty(\Sigma)$ then $X^*, X^\omega \in \mathcal{R}^\infty(\Sigma)$

# $\omega$-Regular Languages

**Theorem 2** *A language* $L \subseteq \Sigma^\omega$ *is* $\omega$*-regular if and only if*

$$L = \bigcup_{i=1}^{n} X_i Y_i^\omega$$

*for some* $X_i, Y_i \in \mathcal{R}(\Sigma)$.

$$\begin{aligned} \mathcal{C} &= \{\bigcup_{i=1}^{n} X_i Y_i^\omega \mid n \geq 0,\ X_i, Y_i \in \mathcal{R}(\Sigma)\} \\ \mathcal{E} &= \{X \in \Sigma^\infty \mid X \cap \Sigma^* \in \mathcal{R}(\Sigma),\ X \cap \Sigma^\omega \in \mathcal{C}\} \end{aligned}$$

We prove that $\mathcal{R}^\infty(\Sigma) \subseteq \mathcal{E}$

# Star Free Languages

The class of *star-free languages* is the smallest class $SF(\Sigma)$ of languages $L \in \Sigma^*$ such that:

- $\emptyset, \{\epsilon\} \in SF(\Sigma)$ and $\{a\} \in SF(\Sigma)$ for all $a \in \Sigma$
- if $X, Y \in SF(\Sigma)$ then $X \cup Y, XY, \overline{X} \in SF(\Sigma)$

***Example 1***

- $\Sigma^* = \overline{\emptyset}$ *is star-free*
- *if* $B \subset \Sigma$, *then* $\Sigma^* B \Sigma^* = \bigcup_{b \in B} \Sigma^* b \Sigma^*$ *is star-free*
- *if* $B \subset \Sigma$, *then* $B^* = \overline{\Sigma^* \overline{B} \Sigma^*}$ *is star-free*
- *if* $\Sigma = \{a, b\}$, *then* $(ab)^* = \overline{b\Sigma^* \cup \Sigma^* a \cup \Sigma^* aa \Sigma^* \cup \Sigma^* bb \Sigma^*}$ *is star-free*

# Star Free $\omega$-Languages

The class of *star-free $\omega$-languages* is the smallest class $SF^{\infty}(\Sigma)$ of languages $L \in \Sigma^*$ such that:

- $\emptyset, \{a\} \in SF^{\infty}(\Sigma),\ a \in \Sigma$
- if $X, Y \in SF^{\infty}(\Sigma)$ then $X \cup Y, \overline{X} \in SF^{\infty}(\Sigma)$
- if $X \subseteq \Sigma^*,\ X \in SF(\Sigma),\ Y \in SF^{\infty}(\Sigma)$ then $XY \in SF^{\infty}(\Sigma)$

***Example 2***

- *if $B \subset \Sigma$, then $\Sigma^* B \Sigma^\omega$ is star-free*
- *if $\Sigma = \{a, b\}$, then $(ab)^\omega = \overline{b\Sigma^\omega \cup \Sigma^* aa\Sigma^\omega \cup \Sigma^* bb\Sigma^\omega}$ is star-free*

# Aperiodic Languages

**Definition 1** *A language $L \subseteq \Sigma^*$ is said to be* aperiodic *iff:*

$$\exists n_0 \forall n \geq n_0 \forall u, v, t \in \Sigma^* \ . \ uv^n t \in L \iff uv^{n+1} t \in L$$

*$n_0$ is called the* index *of $L$.*

***Example 3*** *$0^*1^*$ is aperiodic. Let $n_0 = 2$. We have three cases:*

1. *$u, v \in 0^*$ and $t \in 0^*1^*$ :*

$$\forall n \geq n_0 \ . \ uv^n t \in L$$

2. *$u \in 0^*$, $v \in 0^*1^*$ and $t \in 1^*$ :*

$$\forall n \geq n_0 \ . \ uv^n t \notin L$$

3. *$u \in 0^*1^*$, $v \in 1^*$ and $t \in 1^*$ :*

$$\forall n \geq n_0 \ . \ uv^n t \in L$$

# Periodic Languages

Conversely, a language $L \subseteq \Sigma^*$ is said to be *periodic* iff:

$$\forall n_0 \exists n \geq n_0 \exists u, v, t \in \Sigma^* \,.\, (uv^n t \notin L \wedge uv^{n+1} t \in L) \vee (uv^n t \in L \wedge uv^{n+1} t \notin L)$$

***Example 4*** $(00)^*1$ *is periodic.*

*Given* $n_0$ *take the next even number* $n \geq n_0$, $u = \epsilon$, $v = 0$ *and* $t = 1$. *Then* $uv^n t \in (00)^*1$ *and* $uv^{n+1} t \notin (00)^*1$. □

***Exercise 1*** *Is the language* $(ab)^*$ *periodic or aperiodic ?*

# Aperiodic Monoids

**Definition 2** *A monoid $M$ is said to be* aperiodic *iff*

$$\exists n_0 \forall n \geq n_0 \forall x \in M \ . \ x^n = x^{n+1}$$

*$n_0$ is called the* index *of $M$.*

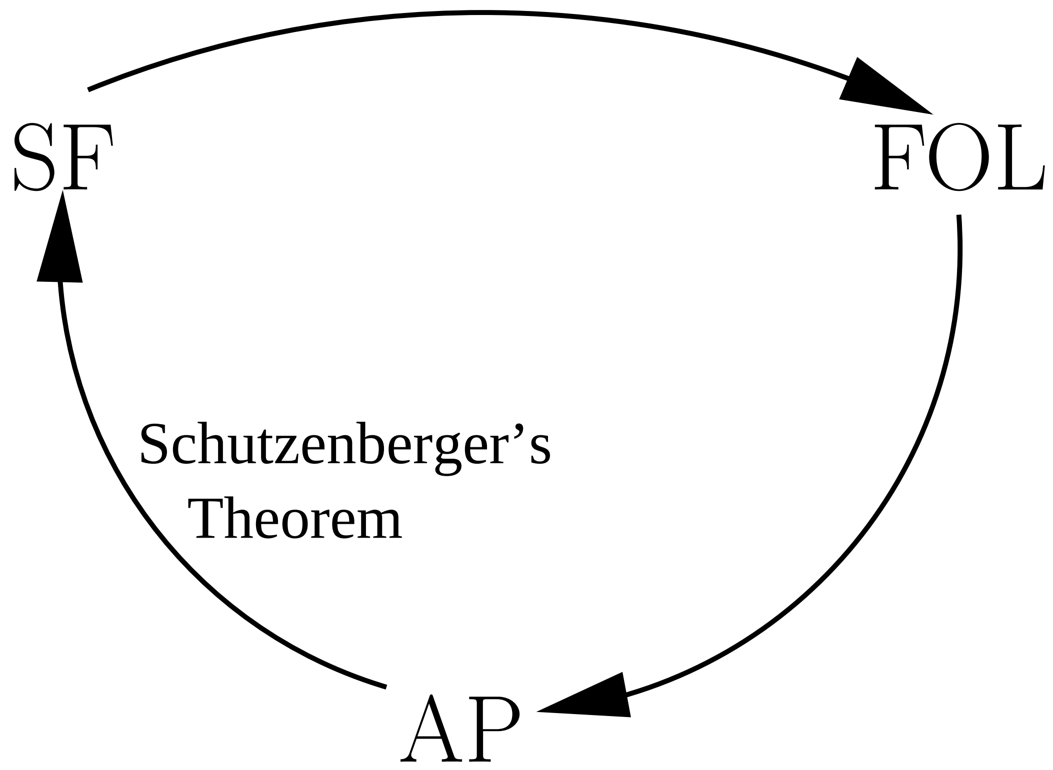
**Proposition 1** *A language $L \subseteq \Sigma^*$ is aperiodic iff its syntactic monoid is aperiodic.*

"$\Rightarrow$" Let $X \in \Sigma^*_{/\simeq_L}$ be an equivalence class of the syntactic monoid of $L$.

If $u \in X$, then $u^n \in X^n$ for all $n \geq 0$. Since $L$ is aperiodic, there exists $n_0$ such that $u^n \simeq_L u^{n+1}$ for all $n \geq n_0$, hence $X^n = X^{n+1}$.

The other direction is similar.

## The Big Picture



# Star Free Languages are FOL-definable

We prove that for each $L \subseteq \Sigma^*$, $L \in SF(\Sigma)$ there exists an FOL sentence $\varphi_L$ such that:

$$L = \{u \in \Sigma^* \mid u \models \varphi_L\}$$

By induction on the structure of $L$:

$$\emptyset = \{u \in \Sigma^* \mid u \models \bot\} \qquad \{a\} = \{u \in \Sigma^* \mid u \models p_a(0) \wedge len(1)\}$$

$$X \cup Y = \{u \in \Sigma^* \mid u \models \varphi_X \vee \varphi_Y\} \qquad \overline{X} = \{u \in \Sigma^* \mid u \models \neg\varphi_X\}$$

$$XY = \exists y \exists z \;.\; 0 \leq y < z \wedge \varphi_X(0,y) \wedge \varphi_Y(y,z) \wedge len(z)$$

where:

- $\varphi(i,j)$ is a formula s.t. $\forall 0 \leq i < j \leq |u| \;.\; u \models \varphi(i,j) \iff u(i,j) \models \varphi$
- $len(x) \equiv \forall y \;.\; s(y) \leq x$

# FOL-definable Languages are Aperiodic

Let $\varphi(x_1, \ldots, x_n)$ be an FOL formula. We denote

$$L^{\varphi}_{i_1,\ldots,i_n} = \{u \in \Sigma^* \mid u \models \varphi(i_1, \ldots, i_n)\}$$

We prove that, for all $u, v, t \in \Sigma^*$, $i_1, \ldots, i_n \in \mathbb{N}$,

$$uv^n t \in L^{\varphi}_{i_1,\ldots,i_n} \iff uv^{n+1} t \in L^{\varphi}_{i'_1,\ldots,i'_n}$$

where, for all $1 \leq k \leq n$:

- $i'_k = i_k$, if $i_k \leq |u| + n \cdot |v|$
- $i'_k = i_k + |v|$, if $i_k > |u| + n \cdot |v|$

By induction on the structure of $\varphi$:

- the cases $x_1 = x_2$ and $x_1 \leq x_2$ are immediate
- $uv^n t \models p_a(i)$ : if $i \leq |u| + n \cdot |v|$ then $(uv^{n+1}t)_i = (uv^n t)_i = a$; if $i > |u| + n \cdot |v|$ then $(uv^{n+1}t)_{i+|v|} = (uv^n t)_i = a$

# FOL-definable Languages are Aperiodic

For all $u, v, t \in \Sigma^*$, $i_1, \ldots, i_n \in \mathbb{N}$,

$$uv^n t \in L^{\varphi}_{i_1,\ldots,i_n} \iff uv^{n+1} t \in L^{\varphi}_{i'_1,\ldots,i'_n}$$

where, for all $1 \leq k \leq n$:

- $i'_k = i_k$, if $i_k \leq |u| + n \cdot |v|$
- $i'_k = i_k + |v|$, if $i_k > |u| + n \cdot |v|$

By induction on the structure of $\varphi$:

- $\varphi_1 \wedge \varphi_2$ : is immediate
- $\neg\varphi$ : $uv^n t \notin L^{\varphi}_{i_1,\ldots,i_n} \iff uv^{n+1} t \notin L^{\varphi}_{i'_1,\ldots,i'_n}$
- $\exists x_1 \,.\, \varphi(x_1, \ldots, x_n)$ : $uv^n t \in L^{\exists x_1 \,.\, \varphi}_{i_2,\ldots,i_n} \iff uv^n t \in L^{\varphi}_{i_1,i_2,\ldots,i_n}$ for some $i_1 \in \mathbb{N}$. By the induction hypothesis, $uv^{n+1} t \in L^{\varphi}_{i'_1,i'_2,\ldots,i'_n}$, hence $uv^{n+1} t \in L^{\exists x_1 \,.\, \varphi}_{i'_2,\ldots,i'_n}$. The other direction is symmetric.

# Schützenberger's Theorem

**Theorem 3** *For any recognizable* $L \subseteq \Sigma^*$*,* $L$ *is star-free iff* $L$ *is aperiodic.*

"$\Rightarrow$" Prove the existence of an integer $N(L)$ such that

$$\forall n \geq N(L)\ \forall u \forall v \forall t\ .\ uv^n t \in L \iff uv^{n+1}t \in L$$

Suppose $v \neq \epsilon$. By induction on the structure of $L$:

- $\emptyset$ : $N(\emptyset) = 0$, since $\forall n \geq 0\ .\ uv^n t \notin L$
- $\{a\}, a \in \Sigma$ : $N(\{a\}) = 2$, since $\forall n \geq 2\ .\ uv^n t \notin L$
- $\overline{X}$ : $N(\overline{X}) = N(X)$, trivial
- $X \cup Y$ : $N(X \cup Y) = \max\{N(X), N(Y)\}$, trivial
- $XY$ : $N(XY) = N(X) + N(Y) + 1$, since for all $n = n_1 + n_2 + 1 \geq N(X) + N(Y) + 1$, we have either $n_1 \geq N(X)$ or $n_2 \geq N(Y)$. Then $uv^n t = (uv^{n_1} r)(sv^{n_2} t)$, where $rs = v$ and $uv^{n_1} r \in X$, $sv^{n_2} t \in Y$. If $n_1 \geq N(X)$, $uv^{n_1+1} r \in X \Rightarrow uv^{n+1} t \in XY$

# Schützenberger's Theorem

"$\Leftarrow$" Let $\varphi : \Sigma^* \to M$ be a monoid morphism, for an aperiodic monoid $M$.

If $L$ is recognizable, then there exists a finite subset $P \subseteq M$ such that

$$L = \varphi^{-1}(P) = \bigcup_{m \in P} \varphi^{-1}(m)$$

We show that $\varphi^{-1}(m)$ is star-free, for each $m \in M$.

# The Simplification Rule

**Lemma 1** *Let $M$ be an aperiodic monoid and let $p, q, r \in M$. If $pqr = q$, then $pq = q = qr$.*

Let $N$ be the index of $M$.

If $pqr = q$ then $p^n qr^n = q$, for all $n > 0$.

If $n \geq N$ then $p^{n+1} qr^n = p^n qr^n = q$.

Hence $p(p^n qr^n) = pq = q$. The proof of $q = qr$ is similar. $\square$

# Examples of using SR

Let $M$ be an aperiodic monoid and $m \in M$.

***Example 5*** *If* $M = mM$ *then* $1 = m \cdot x$, *for some* $x \in M$.

$$\begin{aligned} 1 &= m \cdot x \\ &= m \cdot 1 \cdot x \\ &\overset{(SR)}{=} m \cdot 1 \;=\; m \end{aligned}$$

***Example 6*** *If* $M = MmM$ *then* $1 = x \cdot m \cdot y$, *for some* $x, y \in M$.

$$\begin{aligned} 1 &= x \cdot m \cdot y \\ &= (x \cdot m) \cdot 1 \cdot y \\ &= (x \cdot m) \cdot 1 \;=\; x \cdot m \\ &= x \cdot (1 \cdot m) \\ &= 1 \cdot m \;=\; m \end{aligned}$$

# Ideals

Let $M$ be a monoid.

A set $R \subseteq M$ is a right ideal of $M$ iff:

$$RM = R \iff \forall r \in R \; \forall x \in M \; . \; rx \in R$$

A set $L \subseteq M$ is a left ideal of $M$ iff:

$$ML = L \iff \forall l \in L \; \forall x \in M \; . \; xl \in L$$

A set $I \subseteq M$ is a ideal of $M$ iff:

$$MIM = I \iff \forall x, y \in M \; \forall i \in I \; . \; xiy \in I$$

# Green Relations

Let $M$ be a monoid and $a, b \in M$. Then we define

$$a \leq_{\mathcal{R}} b \iff aM \subseteq bM$$

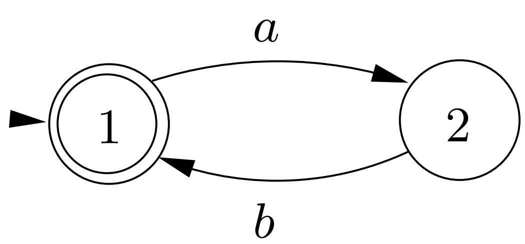
$$a \leq_{\mathcal{L}} b \iff Ma \subseteq Mb$$

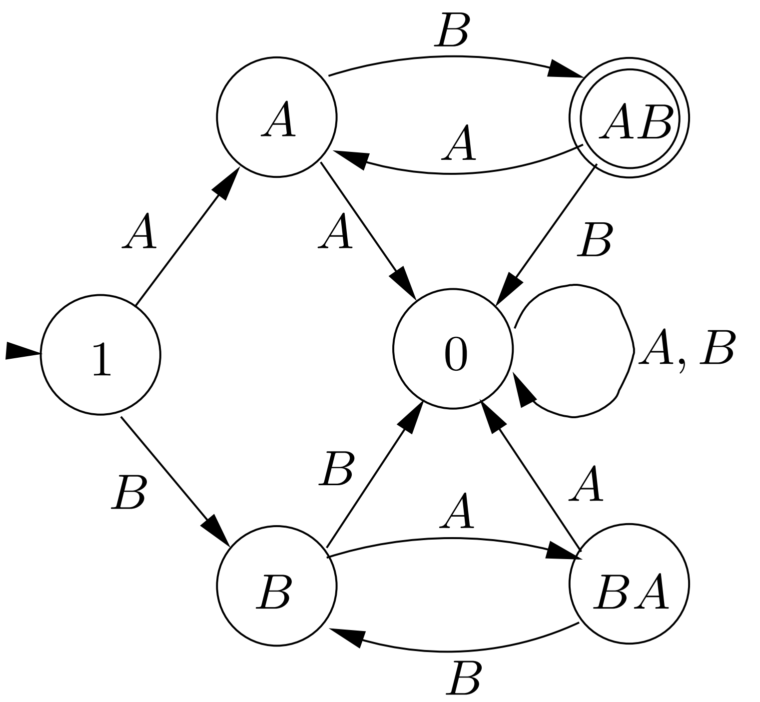
$$a \,\mathcal{R}\, b \iff aM = bM$$

$$a \,\mathcal{L}\, b \iff Ma = Mb$$

$$a \,\mathcal{I}\, b \iff MaM = MbM$$

# Example




$$1M = M \qquad M1 = M \qquad M1M = M$$

$$0M = \{0\} \qquad M0 = \{0\} \qquad M0M = \{0\}$$

$$ABM = AM = \{A, AB, 0\} \qquad MBA = MA = \{A, BA, 0\}$$

$$BAM = BM = \{B, BA, 0\} \qquad MAB = MB = \{B, AB, 0\}$$

$$MAM = MBM = MABM = MBAM = \{A, BA, AB, 0\}$$

# A Consequence

**Lemma 2** *Let $M$ be an aperiodic monoid and let $a, b \in M$ such that $a \,\mathcal{I}\, b$. Then we have:*

- $a \leq_{\mathcal{R}} b \;\Rightarrow\; a \,\mathcal{R}\, b$
- $a \leq_{\mathcal{L}} b \;\Rightarrow\; a \,\mathcal{L}\, b$

$MaM = MbM \;\Rightarrow\; b = uav$, for some $u, v \in M$

$aM \subseteq bM \;\Rightarrow\; a = bp$, for some $p \in M$

$b = uav = ubpv \overset{(SR)}{=} bpv$

$bM = bpvM \subseteq bpM = aM$, hence $aM = bM$. $\square$

# Decomposition in Ideals

**Lemma 3** *Let $M$ be an aperiodic monoid and let $p, q, r \in M$. Then* $\{q\} = (qM \cap Mq) \setminus J_q$, *with* $J_q = \{s \in M \mid q \notin MsM\}$.

$q \in (qM \cap Mq) \setminus J_q$ is trivial.

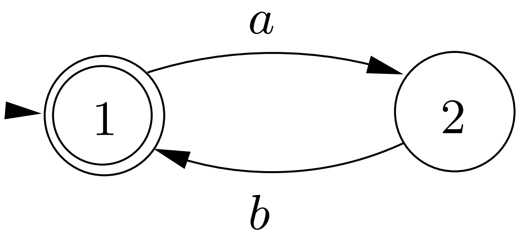
$s \in (qM \cap Mq) \setminus J_q \;\Rightarrow\; \exists p, r \in M \;.\; pq = qr = s$

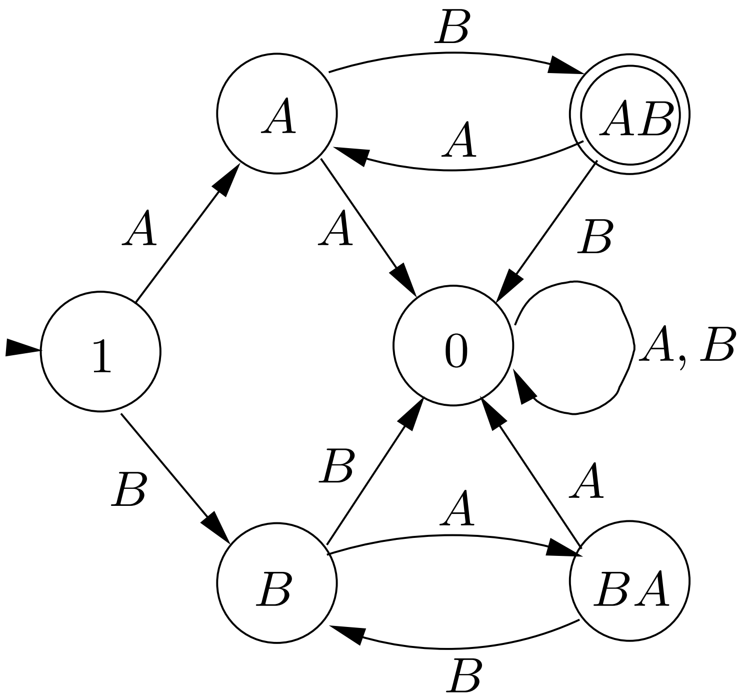
$s \notin J_q \;\Rightarrow\; q \in MsM \;\Rightarrow\; \exists u, v \in M \;.\; usv = q$

$q = usv = u(pq)v = (up)q(v) \overset{(SR)}{=} (up)q = u(pq) = us$

$q = us = u(qr) \overset{(SR)}{=} qr = s$ □

# Example




$$\begin{aligned} AM &= \{A, AB, 0\} \\ MA &= \{A, BA, 0\} \\ J_A &= \{0\} \\ \{A\} &= (AM \cap MA) \setminus J_A \end{aligned}$$

# Schützenberger's Theorem (Base Case)

We prove $\varphi^{-1}(m)$ is star-free by induction on $r(m) = \|M \setminus MmM\|$

The base case $r(m) = 0$. Then $M = MmM$, hence $m = 1$ (SR).

We show $\varphi^{-1}(1) = (\{a \in \Sigma \mid \varphi(a) = 1\})^*$:

$$\begin{aligned} \varphi(ua) &= 1 \\ \varphi(u)1\varphi(a) &= 1 \\ 1\varphi(a) &= 1 \ (SR) \\ \varphi(a) &= 1 \end{aligned}$$

$B^* = \overline{\Sigma^* \overline{B} \Sigma^*}$, for any $B \subseteq \Sigma$ is star-free.

# Schützenberger's Theorem (Induction Step)

We show that

$$\varphi^{-1}(m) = (U\Sigma^* \cap \Sigma^* V) \setminus (\Sigma^* C \Sigma^* \cup \Sigma^* W \Sigma^*)$$

where

$$U = \bigcup_{(n,a)\in E} \varphi^{-1}(n)a\Sigma^* \qquad V = \bigcup_{(a,n)\in F} \Sigma^* a \varphi^{-1}(n)$$

$$C = \{a \in \Sigma \mid m \notin M\varphi(a)M\} \quad W = \bigcup_{(a,n,b)\in G} \Sigma^* a \varphi^{-1}(n) b \Sigma^*$$

with

$$E = \{(n,a) \in M \times \Sigma \mid n\varphi(a) \ \mathcal{R} \ m, \ n \notin mM\}$$

$$F = \{(a,n) \in \Sigma \times M \mid \varphi(a)n \ \mathcal{L} \ m, \ n \notin Mm)\}$$

$$G = \{(a,n,b) \in \Sigma \times M \times \Sigma \mid m \in (M\varphi(a)nM \cap Mn\varphi(b)M) \setminus M\varphi(a)n\varphi(b)M\}$$

Then it is enough to show that $U, V, C$ and $W$ are star-free languages.

## $\varphi^{-1}(m) \subseteq L$ (1)

W.l.o.g. suppose $m \neq 1$. Let $u \in \varphi^{-1}(m)$ and show $u \in U\Sigma^*$, where

$$\begin{aligned} U &= \bigcup_{(n,a)\in E} \varphi^{-1}(n)a\Sigma^* \\ E &= \{(n,a) \in M \times \Sigma \mid n\varphi(a) \mathrel{\mathcal{R}} m,\ n \notin mM\} \end{aligned}$$

Let $p$ be the shortest prefix of $u$ such that $\varphi(p) \mathrel{\mathcal{R}} m$.

If $p = \epsilon$ then $1 \mathrel{\mathcal{R}} m \Rightarrow m = 1$. Hence $p \neq \epsilon$, and let $p = ra$, $a \in \Sigma$.

Let $n = \varphi(r)$. We have $n\varphi(a) \mathrel{\mathcal{R}} m$ and not $n \mathrel{\mathcal{R}} m$ (choice of $p$), hence $(n,a) \in E \Rightarrow u \in U\Sigma^*$. Symmetric argument for $u \in \Sigma^*V$.

## $\varphi^{-1}(m) \subseteq L$ **(2)**

Let $u \in \varphi^{-1}(m)$ and show $u \notin \Sigma^* C \Sigma^*$, where

$$C = \{a \in \Sigma \mid m \notin M\varphi(a)M\}$$

Suppose $u \in \Sigma^* C \Sigma^*$, then $\varphi(u) = m \in M\varphi(a)M$, for some $a \in C$, contradiction.

We show $u \notin \Sigma^* W \Sigma^*$, where $W = \bigcup_{(a,n,b) \in G} \Sigma^* a \varphi^{-1}(n) b \Sigma^*$ and

$G = \{(a, n, b) \in \Sigma \times M \times \Sigma \mid m \in (M\varphi(a)nM \cap Mn\varphi(b)M) \setminus M\varphi(a)n\varphi(b)M\}$

If $u \in \Sigma^* a \varphi^{-1}(n) b \Sigma^*$ for some $(a, n, b) \in G$ then
$\varphi(u) = m \in M\varphi(a)n\varphi(b)M$, contradiction.

## $L \subseteq \varphi^{-1}(m)$

Let $u \in L = (U\Sigma^* \cap \Sigma^* V) \setminus (\Sigma^* C \Sigma^* \cup \Sigma^* W \Sigma^*)$ and $s = \varphi(u)$.

$u \in U\Sigma^* = \bigcup_{(n,a)\in E} \varphi^{-1}(n) a \Sigma^* \;\Rightarrow\; s \in n\varphi(a)M$ for some $(n, a) \in E \Rightarrow n\varphi(a) \; \mathcal{R} \; m$, hence $s \in mM$. Symmetrically, $s \in Mm$.

$\{s\} = \{m\} \overset{(ID)}{=} (mM \cap Mm) \setminus J_m \;\Leftarrow\; s \notin J_m \iff m \in MsM$

Let $f$ be the smallest factor of $u$ s.t. $m \notin M\varphi(f)M$.

- $f = \epsilon \;\Rightarrow\; m \notin M$, contradiction.
- $f = a \in \Sigma \;\Rightarrow\; a \in C \;\Rightarrow\; u \in \Sigma^* C \Sigma^*$, contradiction.
- $f = agb,\; a, b \in \Sigma$ and let $n = \varphi(g)$ and $\varphi(f) = \varphi(a)n\varphi(b)$. Then $m \in M\varphi(a)nM \cap Mn\varphi(b)M \;\Rightarrow\; (a, n, b) \in G \;\Rightarrow\; f \in W \;\Rightarrow u \in \Sigma^* W \Sigma^*$, contradiction.

We conclude that $m \in M\varphi(u)M = MsM$.

# Schützenberger's Theorem (Induction Step)

$$\boxed{\varphi^{-1}(m) \text{ is star-free for each } r(m) < n \Rightarrow \varphi^{-1}(m) \text{ is star-free for } r(m) = n}$$

Since $\varphi^{-1}(m) = (U\Sigma^* \cap \Sigma^* V) \setminus (\Sigma^* C \Sigma^* \cup \Sigma^* W \Sigma^*)$, it is enough to show that $U, V$ and $W$ are star-free ($\Sigma^* C \Sigma^*$ is trivially star-free)

$$U = \bigcup_{(n,a)\in E} \varphi^{-1}(n) a \Sigma^*, \;\; E = \{(n,a) \in M \times \Sigma \mid n\varphi(a) \;\mathcal{R}\; m, \;\; n \notin mM\}$$

$$(n,a) \in E \Rightarrow n\varphi(a) \;\mathcal{R}\; m \Rightarrow Mn\varphi(a)M = MmM \subseteq MnM \Rightarrow r(n) \leq r(m)$$

Suppose $r(n) = r(m) \iff n \;\mathcal{J}\; m \overset{m \leq_{\mathcal{R}} n}{\Longrightarrow} n \;\mathcal{R}\; m$, contradiction. Hence $r(n) < r(m) \Rightarrow \varphi^{-1}(n)$ is star-free $\Rightarrow$ $U$ is star-free. Symmetric for $V$.

# Schützenberger's Theorem (Induction Step)

$W = \bigcup_{(a,n,b)\in G} \Sigma^* a\varphi^{-1}(n) b\Sigma^*$

$G = \{(a,n,b) \in \Sigma \times M \times \Sigma \mid m \in (M\varphi(a)nM \cap Mn\varphi(b)M) \setminus M\varphi(a)n\varphi(b)M\}$

$(a,n,b) \in G \Rightarrow m \in (M\varphi(a)nM \cap Mn\varphi(b)M) \setminus M\varphi(a)n\varphi(b)M)$

$m \in M\varphi(a)nM \subseteq MnM \Rightarrow r(n) \leq r(m)$

Suppose $r(n) = r(m) \Rightarrow MnM = MmM \Rightarrow n \in MmM$. Since $m \in Mn\varphi(b)M$, we have $n \in Mn\varphi(b)M \Rightarrow n = un\varphi(b)v \stackrel{SR}{\Rightarrow} n = n\varphi(b)v$

$m \in M\varphi(a)nM \Rightarrow m = x\varphi(a)ny = x\varphi(a)n\varphi(b)vy$, contradiction with $m \notin M\varphi(a)n\varphi(b)M$

Hence $r(n) < r(m) \Rightarrow W$ is star-free. □

## Example

$$1M = M \qquad M1 = M \qquad M1M = M$$

$$0M = \{0\} \qquad M0 = \{0\} \qquad M0M = \{0\}$$

$$ABM = AM = \{A, AB, 0\} \qquad MBA = MA = \{A, BA, 0\}$$

$$BAM = BM = \{B, BA, 0\} \qquad MAB = MB = \{B, AB, 0\}$$

$$MAM = MBM = MABM = MBAM = \{A, BA, AB, 0\}$$

Let us compute $\varphi^{-1}(AB)$:

$$1A \mathrel{\mathcal{R}} AB,\ 1 \notin ABM \quad \Rightarrow \quad E = \{(1, a)\} \quad \Rightarrow \quad U\Sigma^* = a\Sigma^*$$

$$B1 \mathrel{\mathcal{L}} AB,\ 1 \notin MAB \quad \Rightarrow \quad F = \{(b, 1)\} \quad \Rightarrow \quad \Sigma^* V = \Sigma^* b$$

$$AB \notin M(A1A)M = M(B1B)M \ \Rightarrow \ G = \{(a, 1, a), (b, 1, b)\}$$

$$\Rightarrow \ \Sigma^* W \Sigma^* = \Sigma^* aa \Sigma^* \cup \Sigma^* bb \Sigma^*$$

$$\boxed{\varphi^{-1}(AB) = (ab)^* = (a\Sigma^* \cap \Sigma^* b) \setminus (\Sigma^* aa \Sigma^* \cup \Sigma^* bb \Sigma^*)}$$

# Equivalence of Star Free, Aperiodic and FOL-definable Languages