Parity Games

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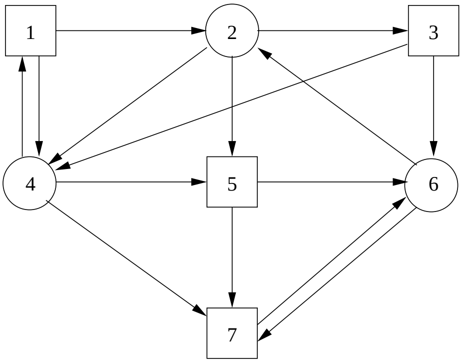
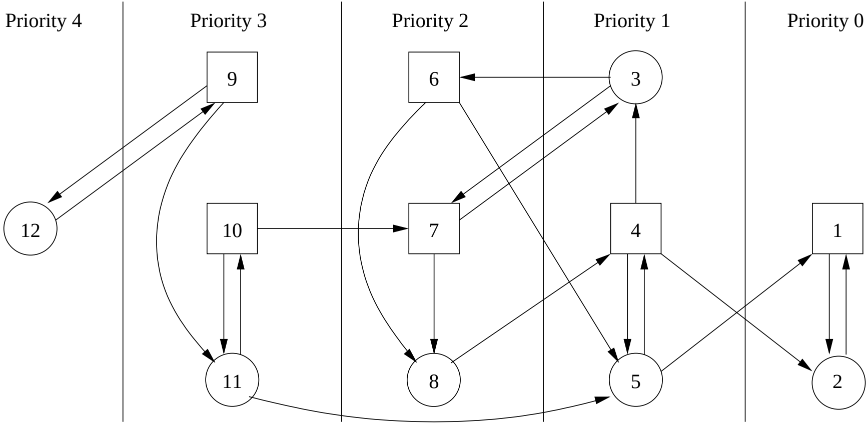
Homework

1. Consider the game graph shown below. Let the winning condition for Player 0 be $\mathrm{Occ}(\rho) = \{1, 2, 3, 4, 5, 6, 7\}$.
    1. Find the winning region for Player 0 and describe a winning strategy
    2. Show that there is no positional winning strategy for Player 0.



Homework

2. Compute the winning regions and the corresponding positional winning strategies for Player 0 and 1 in this weak-parity game.



Homework

3. A winning strategy is called *uniform* if it is a winning strategy from every winning state in the game. Let (G, p) be a weak parity game and let W_0 be the winning region of Player 0. For all $s \in W_0$ let f_s be a positional winning strategy from s for Player 0. Construct a uniform winning strategy f from the strategies f_s meaning that for every $s \in W_0$ there is a $t \in W_0$, s.t. $f(s) = f_t(s)$.

Parity Games

A Parity game is a pair (G, p), where

- $G = (S, S_0, E)$ is a game graph and
- $p : S \to \{0, \ldots, k\}$ is a priority function mapping every state in S to a number in $\{0, \ldots, k\}$.

A play ρ is winning for Player 0 iff the minimum priority visited infinitely often in ρ is even: $\mathsf{min}_{s \in \mathrm{Inf}(\rho)} p(s)$ is even.

Parity Games

Theorem

1. *Parity games are determined (i.e., each state belongs to* W_0 *or* W_1*), and the winner from a given state has a positional winning strategy.*
2. *Over finite graphs, the winning regions and winning strategies of the two players can be computed in (at most) exponential time in the number of vertices of the game graph.*

Proof

Given $G = (S, S_0, E)$ with priority function $p : S \to \{0, \ldots, k\}$. We proceed by induction on the number of states denoted by n.

- Basis case: we either have one Player-0 or Player-1 state with a selfloop (Note that every state in a game has at least one outgoing edge). Then the priority of the state determines if $S = W_0$ or $S = W_1$.
- Induction step: Let $P_i = \{s \mid p(s) = i\}$ be the set of states with priority i. Assume $P_0 \neq \emptyset$, otherwise assume $P_1 \neq \emptyset$ and switch the roles of Players 0 and 1 below. Finally, if $P_0 = P_1 = \emptyset$ decrease every priority by 2.

Proof (induction step cont.)

Choose $s \in P_0$ and let $X = \mathrm{Attr}_0(\{s\})$. Note that $S \setminus X$ is a subgame with $< n$ states.

The induction hypothesis gives a partition of $S \setminus X$ into winning regions U_0 and U_1 for Player 0 and 1, respectively, and corresponding positional winning strategies.

- Case 1: Player 0 can guarantee a transition from s to $U_0 \cup X$, i.e., if $s \in S_0$, then there exists $s' \in U_0 \cup X$ such that $(s, s') \in E$ or if $s \in S_1$, then for all $(s, s') \in E$, $s' \in U_0 \cup X$ holds.

  Claim:

  (i) $U_0 \cup X \subseteq W_0$

  (ii) $U_1 \subseteq W_1$.

Proof (Case 1 cont.)

The positional strategy for Player 0 on $U_0 \cup X$ is:

1. On U_0 play according to the positional strategy given by the induction hypothesis
2. On X $(= \mathrm{Attr}_0(\{s\}))$ play according to the attractor strategy. Then eventually reach s
3. From s "move back" to $U_0 \cup X$.

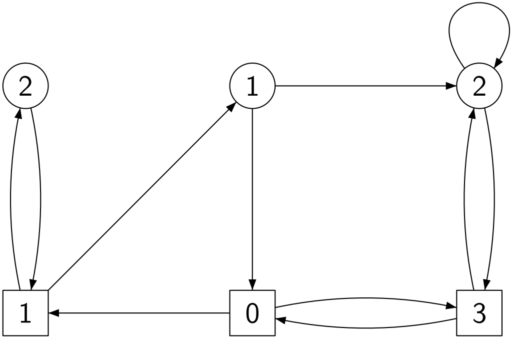
For Player 1 use the positional strategy on U_1 given by the induction hypothesis.

Proof of claim: (ii) is clear, since starting in U_1 Player 1 can guarantee that the play remains in U_1 (see picture). For (i), the play remains in $U_0 \cup X$ if the strategy for state s is followed. If the play eventually remains in U_0, then Player 0 wins by induction hypothesis, otherwise the play passes through s infinitely often, which is winning as well.

Proof (Case 2)

- Case 2: Player 1 can guarantee a transition to U_1 from s, i.e., if $s \in S_0$, then all edges $(s, s') \in E$ lead to U_1 ($s' \in U_1$), and if $s \in S_1$, then there exists $s' \in U_1$ such that $(s, s') \in E$.
  Let $Y = \mathrm{Attr}_1(U_1)$, then $s \in Y$ and $S \setminus Y$ is a subgame with $< n$ states. The induction hypothesis gives winning region V_0 and V_1 and corresponding positional winning strategies.
  Claim:
  (i) $V_0 \subseteq W_0$
  (ii) $V_1 \cup Y \subseteq W_1$.
  Proof of claim: (i) is clear, since Player 0 can guarantee to stay within V_0. For (ii), for all states in Y, Player 1 can guarantee to move to U_1 and remind there. From $t \in V_1$ Player 0 can either move to Y or stay in V_1. Both choices are winning for Player 1.

Example



Complexity

Solve(G)= $\hfill T(n)$

1. Pick s + (U_0, U_1)=Solve($G \setminus \text{Attr}_*(\{s\})$ $\quad O(m) + T(n-1)$

2.a If s has edge to $U_* \cup \text{Attr}_*(\{s\})$ then DONE

2.b else Solve($G \setminus \text{Attr}_*(U_*)$ $\quad T(n-1)$

Recurrence relation for time complexity:

$$T(n) \leq O(m) + 2 \cdot T(n-1)$$

Hence, $T(n) = O(m \cdot 2^n)$.

A more careful analysis give: $T(n) = O((\frac{n}{d})^d)$

Note that the exact complexity class of parity games is still an open question.

Next, we show that parity games are in NP $\cap$ co-NP.

Uniform Positional Strategies

Theorem

Given a parity game over $G = (S, S_0, E)$, there is a single positional strategy f such that from each $s \in W_0$ the strategy f is a winning strategy for Player 0 from s.

Proof.

Number the states by natural numbers. Denote by s_i the state with number i. For $s_i \in W_0$ choose a corresponding positional winning strategy f_i. Let F_i be the set of reachable states by plays from s_i according to f_i (Note: $F_i \subseteq W_0$ and $s_i \in F_i$)

Merging Strategies

Define f on W_0 as follows: $f(s) = f_i(s)$ for the smallest i such that $s \in F_i$.

Show that f is a winning strategy from any $s \in W_0$.

Applying f during a play means to apply strategies f_i where i is weakly decreasing. From some point k onwards, index i stays constant (at the latest when $i = 0$), i.e. the f-values coincide with the f_i-values. The highest colour occurring infinitely often in the play is thus determined by the fixed strategy f_i.

Since f_i is a winning strategy, Player 0 wins the play.

Parity Games are in NP ∩ co-NP

Given a game (G, p) with $G = (S, S_0, E)$ and $p : S \rightarrow \{0, \ldots d\}$, decide if $s \in W_0$.

- First, guess a uniform strategy f for Player 0 (= a set of Player-0 edges $\rightarrow$ polynomial size)
- Restrict the game to f
- Check if f is a winning strategy from s. This can be done in polynomial time as follows: forall odd $i \in \{0, \ldots, d\}$, consider the graph with the states $\bigcup_{j=i\ldots d} P_j$, compute the SCC and check if there exists a SCC C s.t. $C \cap P_i \neq \emptyset$ (meaning that there exists a strategy for Player 1 to force a cycle with an odd minimal priority $\rightarrow f$ is not winning).

Small Parity Progress Measure Algorithm

- Idea: for each state count how many visits Player 1 can force to an odd priority, without visiting a lower even priority.
- Notation:
  - we will use tuples $\vec{v} \in \mathbb{N}^d$ of natural numbers as our counters, each component represents one priority.
  - Given two tuples $\vec{v}$ and $\vec{w}$, we use the lexicographic order for the comparision symbols $<, \leq, =, \neq, \geq, >$, e.g., $(1, 0, 3) < (1, 1, 4)$.
  - We will also use truncated versions $<_i, \leq_i, =_i, \neq_i, \geq_i, >_i$, they denote the lexicographic ordering on $\mathbb{N}^i$ applied to the first i components, e.g., $(2, 3, 0) >_2 (2, 2, 4)$ but $(2, 3, 0) =_0 (2, 2, 4)$.

Small Parity Progress Measure

Definition

Let $((S, S_0, E), p)$ be a parity game with $p : S \to \{0, \dots, d-1\}$. A function $g : S \to \mathbb{N}^d$ is a parity progress measure if for all $(s, s') \in E$,

- $g(s) \geq_{p(s)} g(s')$ and
- $g(s) >_{p(s)} g(s')$ if $p(s)$ is odd, holds.

Remark: If there is a parity progress measure for a parity graph G then all cycles in G have an even minimal priority.

Proof of remark: Let $g : S \to \mathbb{N}^d$ be a parity progress measure for G. Suppose that there is an odd cycle $s_1, s_2, \dots, s_l$ in G, and let $i = p(s_1)$ be the smallest priority on this cycle. Then, by the definition of progress measure we have $g(s_1) >_i g(s_2) \geq_i \cdots \geq_i g(s_l) \geq_i g(s_1)$, and hence $g(s_1) >_i g(s_1)$ contradicting the assumption.

Small Parity Progress Measure

Let (G, p) be a parity game and let $P_i = \{s \in S \mid p(s) = i\}$ be the set of states with priority $i \in \{0, \ldots, d-1\}$.
We define $M_G \subset \mathbb{N}^d$ as

$$M_G = \{0,1\} \times \{0,1,\ldots |P_1| + 1\} \times \{0,1\} \times \cdots \times \{0,1,\ldots |P_{d-1}| + 1\}$$

Theorem

If all cycles in a parity graph G are even then there is a parity progress measure solving $g : S \to M_G$ for G.

Proof.

We prove the theorem by induction on $|S|$. (In order to be successful with an inductive proof, we add the claim that if $p(s)$ is odd, then $g(s) >_{p(s)} (0, \ldots, 0)$.)

- Base case: if $|S| = 1$, the theorem holds trivially

Small Parity Progress Measure

- Induction step:
  - Assume $P_0 \neq \emptyset$. By induction hypothesis there is a parity progress measure $g : S \setminus P_0 \rightarrow M_G$ for the game graph with states $S \setminus P_0$. Setting $g(s) = (0, .., 0) \in M_G$, for all $s \in P_0$, we get a parity progress measure for G.
  - Assume $P_0 = \emptyset$ and $P_1 \neq \emptyset$. We claim that is a non-trivial partition (W_1, W_2) of S, s.t. there is no edges from W_1 to W_2. Let $u \in P_1$ and define $U \subseteq S$ be the states to which there is a non-trivial path from u. If $U = \emptyset$, then $W_1 = \{u\}$ and $W_2 = S \setminus \{u\}$ is a desired partition, otherwise let $W_1 = U$ and $W_2 = S \setminus U$. W_2 is not empty because $u \notin U$ (otherwise there would be an odd cycle).

Small Parity Progress Measure

- (Cont.) By induction we get the parity progress measures g_1 and g_2 for the subgraph $S \cap W_1$ and $S \cap W_2$. From $|P_i| = |P_i \cap W_1| + |P_i \cap W_2|$ and the additional claim, it follows that $g = g_1 \cup (g_2 + (0, |P_1 \cap W_1|, 0, |P_3 \cap W_1|, \dots)$ is a desire progress measure.
- Assume $P_0 = P_1 = \emptyset$, reduce all priorities by 2.

Game Parity Progress Measure

Let M_G^T be the set $M_G \cup \{\top\}$, in which $\top$ is defined to be the largest element in the lexicographic order. We denote by $M(g, s, s')$ the least $m \in M_G^T$ such that

- $m \geq_{p(s)} g(s')$ and
- $m >_{p(s)} g(s')$ if $p(s)$ is odd or $m = g(s') = \top$

Definition

A function $g : S \to M_G^\top$ is a game parity progress measure if for all $s \in S$, we have

- if $s \in S_0$, then there exists $(s, s') \in E$ s.t. $g(s) \geq_{p(s)} M(g, s, s')$,
- if $s \in S_1$, then for all $(s, s') \in E$, we have $g(s) \geq_{p(s)} M(g, s, s')$.

We denote by $||g||$ the set $\{s \in S \mid g(s) \neq \top\}$.

Small Parity Progress Measure

For every game parity progress measure g, we define a strategy $\tilde{g} : S_0 \to S$ for Player 0 by setting $\tilde{g}(s)$ to be a successor s' with a minimal $g(s')$.

Theorem

If g is a game parity progress measure then $\tilde{g}$ is a winning strategy for Player 0 from $||g||$.

Proof.

Note g is a parity progress measure on $||g||$. Hence, all simple cylces in $S \cap ||g||$ are even. It also follows from definition of a game parity progress measure that $\tilde{g}$ refers only to states in $||g||$. □

Small Parity Progress Measure

Theorem

There is a game progress measure $g : S \to M_G^\top$ such that $||g||$ is the winning region W_0 of Player 0.

Proof.

We know that there is a winning strategy f for Player 0 from her winning region, s.t. all cycles in G_f are even, hence, there is a parity progress measure $g : W_0 \to M_G$ on the game graph with state W_0. It follows that setting $g(s) = \top$ for all $s \in S \setminus W_0$ makes g a game parity progress measure. $\square$

Small Parity Progress Measure

First, we define an ordering and a family of Lift$(\cdot, s)$ operators on the set of functions $S \to M_G^\top$. Given two functions $g, g' : S \to M_G^\top$, we define $g \leq g'$ if $g(s) \leq s(s')$ for all $s \in S$ and $g < g'$ if $g \leq g'$ and $g \neq g'$. (The order defines a complete lattice).

$$Lift(g,s)(t) = \begin{cases} g(t) & \text{if } s \neq t \\ \mathsf{max}\{g(s), \mathsf{min}_{(s,s')\in E} M(g,s,s')\} & \text{if } s = t \in S_0 \\ \mathsf{max}\{g(s), \mathsf{max}_{(s,s')\in E} M(g,s,s')\} & \text{if } s = t \in S_1 \end{cases}$$

Note that the following propositions follow immediately from the definitions of game parity progress measure.

(1) For every $s \in S$, the operator Lift$(\cdot, s)$ is $\leq$-monotone.

(2) A function $g : S \to M_G^\top$ is a game parity progess measure iff Lift$(g, s) \leq g$ for all $s \in S$.

Small Parity Progress Measure

Finally, a simple fixpoint algorithm:

$$g := \lambda s \in S.(0, \ldots, 0)$$

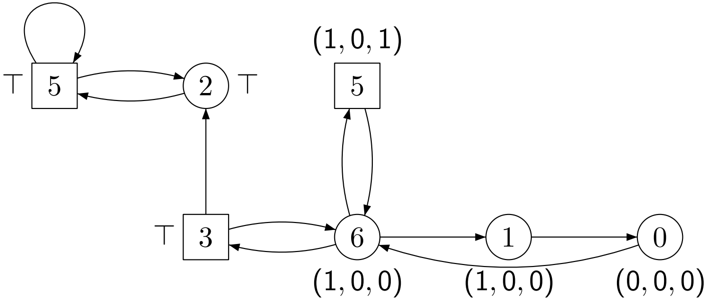
$$\texttt{while } g < \mathrm{Lift}(g, s) \texttt{ for some } s \in S \texttt{ do}$$

$$\quad g := \mathrm{Lift}(g, s)$$

Complexity [Jurdzinski 2000]:

The algorithm runs in $O(dn)$ space and $O(dm \cdot (\frac{n}{\mathrm{floor}(d/2)})^{\mathrm{floor}(d/2)})$ time.

Example+Final Progress Measure



Strategy Improvement

Preparation:

Recall, if players 0 and 1 fix positional strategies f and g, then from each state s a play $G_{f,g}$ is fixed and the winner depends on values in the loop.

Idea: Determine a value $v(s)$ based on $G_{f,g}$

Here v is a valuation function $v : S \to D$ into some value domain D, which is ordered by a preference order.

Format of Strategy Improvement

Given: Priority game graph G, valuation function v

1. Pick two strategies f, g for Players 0 and 1
2. Determine the values $v(s)$ for all $s \in S$, referring to the plays $G_{f,g}$
3. Change strategy f of Player 0 by local improvement: For each S_0-state, choose the out-edge leading to the neighbour states with highest value (by preference order)
4. Given the new f find the optimal response strategy of Player 1 and use it as new strategy g
5. If the new strategies coincide with the previous strategies, then stop; otherwise go back to 2.

Play Profiles (Vöge, Jurdzinski)

Assumption: The states are numbered, and the numbers are the priorities.

Preference order $\prec$ for states $1, \ldots, 8$:

$$1 \prec 3 \prec 5 \prec 8 \prec 6 \prec 4 \prec 2 \prec 0$$

Terminology: The most relevant state of $G_{f,g}$ is the state with the lowest priority in the loop of $G_{f,g}$.

The play profile of $G_{f,g}$ starting from s is the triple (r, P, d) with

- r is the most relevant state of $G_{f,g}$
- P is the set of lower valued states on the path from s to (and excluding) r
- d is the distance between s and r on this path

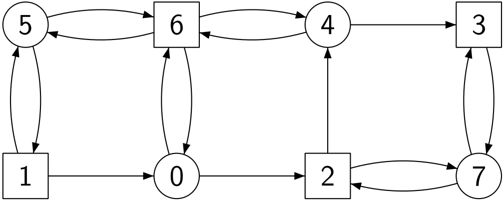
Comparison of Play Profiles

The Preference order is extended from states to play profiles:

$(r, P, d) \prec (r', P', d')$ iff

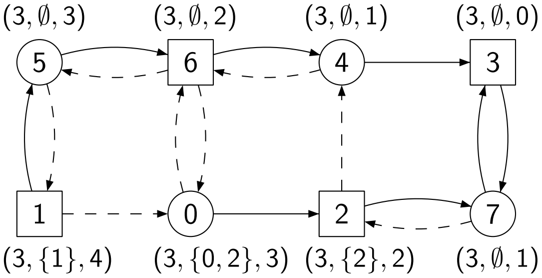
1. $r \prec r'$, or
2. $r = r'$ and the lowest state in the symmetric difference of P, P' is even and belongs to P', or it is odd and belongs to P, or
3. $r = r'$ and $P = P'$ and $d < d'$ if r is odd, or $d' < d$ if r is even.

Example



$f_0, g_0 : 1 \rightarrow 5,\ 5 \rightarrow 6,\ 6 \rightarrow 4,\ 4 \rightarrow 3,\ 3 \rightarrow 7,\ 7 \rightarrow 3,\ 0 \rightarrow 2,\ 2 \rightarrow 7$

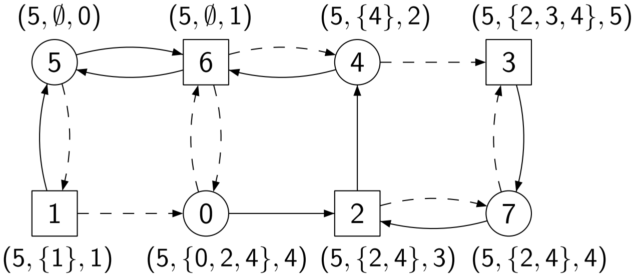
Example



Improve f: 4 → 6 and 7 → 2

Best counterstrategy: 1 → 5, 6 → 5, 2 → 4, 3 → 7.

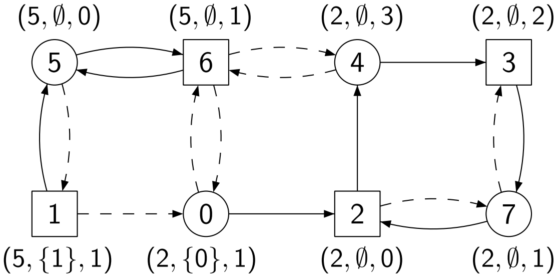
Example



Improve: $4 \rightarrow 3$

Best counterstrategy does not change.

Example



$W_0 = \{0, 2, 3, 4, 7\}$

$W_1 = \{1, 5, 6\}$

Theorem (Vöge, Jurdzinski)

With the valuation by play profiles, the strategy algorithm terminates producing strategies f and g for Players 0 and 1 such that

- $s \in W_0$ ($s \in W_1$) *iff the play* $G_{f,g}$ *ends in a loop with even (respectively, odd) lowest state*
- f *and* g *are winning strategies for Player 0, respectively 1, from the states in* W_0*, respectively* W_1.

Complexity Properties:

- Each improvement round costs polynomial time
- The number of improvement steps is bounded by the number of possible strategies
- Overall improvement steps?