Scaling Dynamic Logic for Intermediate states

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Introduction

Standard sequential program logics are insufficient for reasoning about nonterminating program runs

We extend a dynamic logic with new modalities to talk about resumptions, a pair of intermediate state and residual program

Expressions in DL

The logic consists of expressions of a two sorts

1. programs $(\alpha, \beta, \gamma, \dots)$
2. formulas $(\varphi, \psi, \dots)$

Syntax of programs

We work with non-deterministic and parallel While language and extend it with a special $cont_i$ statement

$$
\begin{aligned}
\mathit{Program}\ p := \quad & \\
& x := a \\
& \textbf{skip} \\
& p; p' \\
& p \mid\mid p' \\
& \textbf{if}\ b\ \textbf{then}\ p_t\ \textbf{else}\ p_f \\
& \textbf{while}\ b\ \textbf{do}\ p \\
& \textbf{cont}_i
\end{aligned}
$$

Syntax of formulas

We introduce two new modalities for reasoning about resumptions

$$\varphi := $$

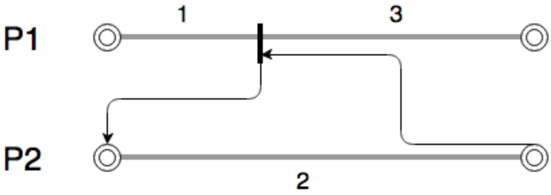
$$x = a$$

$$\varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid 0 \mid 1$$

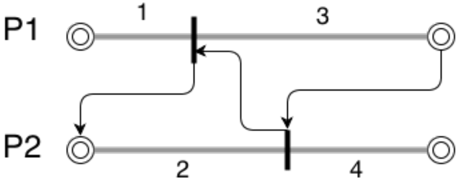
$$\langle p \rangle \varphi \mid [p]\varphi \mid \langle\!\langle p \rangle\!\rangle \varphi \mid [\![p]\!]\varphi$$

Example

$$\langle\langle p1 \rangle\rangle \langle p2 \rangle \langle \mathbf{cont}_0 \rangle \varphi$$



$$[\![p1]\!]\langle\!\langle p2\rangle\!\rangle\langle\mathbf{cont}_1\rangle[\mathbf{cont}_0]\varphi$$



Semantics

$\langle S, \sigma \rangle \rightarrow_{cs} \langle \sigma' \rangle$: states that running S under continuation stack cs from initial state σ terminate in the final state σ'

$\langle S, \sigma \rangle \Rightarrow_{cs} \langle S', \sigma' \rangle$: states that running S under continuation stack cs from initial state σ reaches an intermediate state σ with residual program S'

Natural semantics for While + **cont**

$$\frac{}{\langle x := a, \sigma\rangle \to_{cs} \langle \sigma[x \to [\![a]\!]\sigma]\rangle}\ ass \qquad \frac{}{\langle \mathbf{skip}, \sigma\rangle \to_{cs} \langle \sigma\rangle}\ skip$$

$$\frac{\langle S_1, \sigma\rangle \to_{cs} \langle \sigma'\rangle \quad \langle S_2', \sigma'\rangle \to_{cs} \langle S_2', \sigma''\rangle}{\langle S_1; S_2, \sigma\rangle \to_{cs} \langle \sigma''\rangle}\ comp2$$

$$\frac{[\![b]\!]\sigma = \mathbf{true} \quad \langle S_1, \sigma\rangle \to_{cs} \langle \sigma'\rangle}{\langle \mathbf{if}\ b\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2, \sigma\rangle \to_{cs} \langle \sigma'\rangle}\ ift \qquad \frac{[\![b]\!]\sigma = \mathbf{false} \quad \langle S_2, \sigma\rangle \to_{cs} \langle \sigma'\rangle}{\langle \mathbf{if}\ b\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2, \sigma\rangle \to_{cs} \langle \sigma'\rangle}\ iff$$

$$\frac{[\![b]\!]\sigma = \mathbf{true} \quad \langle S, \sigma\rangle \to_{cs} \langle \sigma'\rangle \quad \langle \mathbf{while}\ b\ \mathbf{do}\ S, \sigma'\rangle \to_{cs} \langle \sigma''\rangle}{\langle \mathbf{while}\ b\ \mathbf{do}\ S, \sigma\rangle \to_{cs} \langle \sigma''\rangle}\ whilet$$

$$\frac{[\![b]\!]\sigma = \mathbf{false}}{\langle \mathbf{while}\ b\ \mathbf{do}\ S, \sigma\rangle \to_{cs} \langle \sigma\rangle}\ whilef$$

$$\frac{\langle cs[i], \sigma\rangle \to_{cs} \langle \sigma'\rangle}{\langle \mathbf{cont}_i, \sigma\rangle \to_{cs} \langle \sigma'\rangle}\ cont$$

Natural semantics for intermediate states excl. final state

$$\frac{}{\langle S,\sigma\rangle \Rightarrow_{cs} \langle S,\sigma\rangle}\ \mathit{init} \quad \text{for ass, if, while}$$

$$\frac{\langle S_1,\sigma\rangle \Rightarrow_{cs} \langle S_1',\sigma'\rangle}{\langle S_1;S_2,\sigma\rangle \Rightarrow_{cs} \langle S_1';S_2,\sigma'\rangle}\ \mathit{comp1}$$

$$\frac{\langle S_1,\sigma\rangle \rightarrow_{cs} \langle \sigma'\rangle \quad \langle S_2,\sigma'\rangle \Rightarrow_{cs} \langle S',\sigma''\rangle}{\langle S_1;S_2,\sigma\rangle \Rightarrow_{cs} \langle S',\sigma''\rangle}\ \mathit{comp2}$$

$$\frac{[\![b]\!]\sigma = \mathbf{true} \quad \langle S_1,\sigma\rangle \Rightarrow_{cs} \langle S_1',\sigma'\rangle}{\langle \mathbf{if}\ b\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2,\sigma\rangle \Rightarrow_{cs} \langle S_1',\sigma'\rangle}\ \mathit{ift} \qquad \frac{[\![b]\!]\sigma = \mathbf{false} \quad \langle S_2,\sigma\rangle \Rightarrow_{cs} \langle S_2',\sigma'\rangle}{\langle \mathbf{if}\ b\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2,\sigma\rangle \Rightarrow_{cs} \langle S_2',\sigma'\rangle}\ \mathit{iff}$$

$$\frac{[\![b]\!]\sigma = \mathbf{true} \quad \langle S,\sigma\rangle \Rightarrow_{cs} \langle S',\sigma'\rangle}{\langle \mathbf{while}\ b\ \mathbf{do}\ S,\sigma\rangle \Rightarrow_{cs} \langle S';\mathbf{while}\ b\ \mathbf{do}\ S,\sigma'\rangle}\ \mathit{while1}$$

$$\frac{[\![b]\!]\sigma = \mathbf{true} \quad \langle S,\sigma\rangle \rightarrow_{cs} \langle \sigma'\rangle \quad \langle \mathbf{while}\ b\ \mathbf{do}\ S,\sigma'\rangle \Rightarrow_{cs} \langle S',\sigma'\rangle}{\langle \mathbf{while}\ b\ \mathbf{do}\ S,\sigma\rangle \Rightarrow_{cs} \langle S',\sigma'\rangle}\ \mathit{while2}$$

$$\frac{\langle cs[i],\sigma\rangle \Rightarrow_{cs} \langle S',\sigma'\rangle}{\langle \mathbf{cont}_i,\sigma\rangle \Rightarrow_{cs} \langle S',\sigma'\rangle}\ \mathit{cont}$$

Final state natural semantics for parallelism

$$\frac{\langle S_1, \sigma\rangle \Rightarrow_{cs} \langle S_1', \sigma'\rangle \quad \langle S_1' || S_2, \sigma'\rangle \rightarrow_{cs} \langle \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \rightarrow_{cs} \langle \sigma''\rangle}\ \mathit{fpara}1$$

$$\frac{\langle S_2, \sigma\rangle \Rightarrow_{cs} \langle S_2', \sigma'\rangle \quad \langle S_1 || S_2', \sigma'\rangle \rightarrow_{cs} \langle \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \rightarrow_{cs} \langle \sigma''\rangle}\ \mathit{fpara}2$$

$$\frac{\langle S_1, \sigma\rangle \rightarrow_{cs} \langle \sigma'\rangle \quad \langle S_2, \sigma'\rangle \rightarrow_{cs} \langle \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \rightarrow_{cs} \langle \sigma''\rangle}\ \mathit{fpara}3$$

$$\frac{\langle S_2, \sigma\rangle \rightarrow_{cs} \langle \sigma'\rangle \quad \langle S_1, \sigma'\rangle \rightarrow_{cs} \langle \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \rightarrow_{cs} \langle \sigma''\rangle}\ \mathit{fpara}4$$

Intermediate state natural semantics for parallelism

$$\frac{\langle S_1, \sigma\rangle \Rightarrow_{cs} \langle S_1', \sigma'\rangle \quad \langle S_1' || S_2, \sigma'\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}\ \mathit{ipara}1$$

$$\frac{\langle S_2, \sigma\rangle \Rightarrow_{cs} \langle S_2', \sigma'\rangle \quad \langle S_1 || S_2', \sigma'\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}\ \mathit{ipara}2$$

$$\frac{\langle S_1, \sigma\rangle \rightarrow_{cs} \langle \sigma'\rangle \quad \langle S_2, \sigma'\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}\ \mathit{ipara}3$$

$$\frac{\langle S_2, \sigma\rangle \rightarrow_{cs} \langle \sigma'\rangle \quad \langle S_1, \sigma'\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}{\langle S_1 || S_2, \sigma\rangle \Rightarrow_{cs} \langle S, \sigma''\rangle}\ \mathit{ipara}4$$

Semantics for modalities

- $\sigma \models_{cs} \langle S \rangle \varphi \iff$

  for some σ' s.t. $S, \sigma \rightarrow_{cs} \sigma'$ we have $\sigma' \models_{cs} \varphi$
- $\sigma \models_{cs} \langle\!\langle S \rangle\!\rangle \varphi \iff$

  for some S', σ' s.t. $S, \sigma \Rightarrow_{cs} S', \sigma'$ we have $\sigma' \models_{S':cs} \varphi$
- $\sigma \models_{cs} [S] \varphi \iff$

  for all σ' s.t. $S, \sigma \rightarrow_{cs} \sigma'$ we have $\sigma' \models_{cs} \varphi$
- $\sigma \models_{cs} [\![S]\!] \varphi \iff$

  for all S', σ' s.t. $S, \sigma \Rightarrow_{cs} S', \sigma'$ we have $\sigma' \models_{S':cs} \varphi$

Substitutions

$$
\begin{aligned}
\mathbf{cont}_i[S/\mathbf{cont}_i] &= S \\
\mathbf{cont}_j[S/\mathbf{cont}_i] &= \mathbf{cont}_j,\ i \neq j \\
x := a[S/\mathbf{cont}_i] &= x := a \\
\mathbf{skip}[S/\mathbf{cont}_i] &= \mathbf{skip} \\
(S'; S'')[S/\mathbf{cont}_i] &= S'[S/\mathbf{cont}_i]; S''[S/cont_i] \\
\mathbf{if}\ b\ \mathbf{then}\ S'\ \mathbf{else}\ S''[S/\mathbf{cont}_i] &= \mathbf{if}\ b\ \mathbf{then}\ S'[S/\mathbf{cont}_i]\ \mathbf{else}\ S''[S/\mathbf{cont}_i] \\
\mathbf{while}\ b\ \mathbf{do}\ S'[S/\mathbf{cont}_i] &= \mathbf{while}\ b\ \mathbf{do}\ S'[S/\mathbf{cont}_i] \\
(\varphi \wedge \psi)[S/\mathbf{cont}_i] &= \varphi[S/\mathbf{cont}_i] \wedge \psi[S/\mathbf{cont}_i] \\
(\varphi \vee \psi)[S/\mathbf{cont}_i] &= \varphi[S/\mathbf{cont}_i] \vee \psi[S/\mathbf{cont}_i] \\
\langle S' \rangle \varphi[S/\mathbf{cont}_i] &= \langle S'[S/\mathbf{cont}_i] \rangle \varphi[S/\mathbf{cont}_i] \\
[S']\varphi[S/\mathbf{cont}_i] &= [S'[S/\mathbf{cont}_i]]\varphi[S/\mathbf{cont}_i] \\
\langle\!\langle S' \rangle\!\rangle \varphi[S/\mathbf{cont}_i] &= \langle\!\langle S'[S/\mathbf{cont}_i] \rangle\!\rangle \varphi[S^{up}/\mathbf{cont}_{i+1}] \\
[\![S']\!] \varphi[S/\mathbf{cont}_i] &= [\![S'[S/\mathbf{cont}_i]]\!] \varphi[S^{up}/\mathbf{cont}_{i+1}]
\end{aligned}
$$

where S^{up} replaces every occurence of $\mathbf{cont}_i$ with $\mathbf{cont}_{i+1}$.

Substitution

Substitution property

$$\sigma \models_{cs} \varphi[S/cont_i] \iff \sigma \models_{cs[i \mapsto S]} \varphi$$

where the notation $cs[i \mapsto S]$ replaces the i-th element in cs by S

Expressivity

$$[\![S]\!]\langle \mathbf{cont}_0 \rangle \varphi - \textit{ should converge}$$

Property: $[\![S]\!]\langle \mathbf{cont}_0 \rangle \varphi \Rightarrow \langle S \rangle \varphi$

$$\langle\!\langle S \rangle\!\rangle [\mathbf{cont}_0] \varphi - \textit{"lucky" intermediate state}$$

Property: $[\![S]\!]\varphi \Rightarrow \langle\!\langle S \rangle\!\rangle [\mathbf{cont}_0] \varphi$

Axioms

$$
\begin{array}{rcl|rcl}
\langle x := a\rangle\varphi & \iff & \varphi[a/x] & \langle\!\langle x := a\rangle\!\rangle\varphi & \iff & \varphi \\
[x := a]\varphi & \iff & \varphi[a/x] & [\![x := a]\!]\varphi & \iff & \varphi \\
\langle\mathbf{skip}\rangle\varphi & \iff & \varphi & \langle\!\langle\mathbf{skip}\rangle\!\rangle\varphi & \iff & \mathit{false} \\
[\mathbf{skip}]\varphi & \iff & \varphi & [\![\mathbf{skip}]\!]\varphi & \iff & \mathit{true} \\
\langle S; S'\rangle\varphi & \iff & \langle S\rangle\langle S'\rangle\varphi & \langle\!\langle S; S'\rangle\!\rangle\varphi & \iff & \langle\!\langle S\rangle\!\rangle(\varphi[\mathbf{cont}_0; S'/\mathbf{cont}_0]) \\
 & & & & & \vee \langle S\rangle\langle\!\langle S'\rangle\!\rangle\varphi \\
[S; S']\varphi & \iff & [S][S']\varphi & [\![S; S']\!]\varphi & \iff & [\![S]\!](\varphi[\mathbf{cont}_0; S'/\mathbf{cont}_0]) \\
 & & & & & \wedge [S][\![S']\!]\varphi
\end{array}
$$

Axioms ctd

$$\langle\!\langle \textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2 \rangle\!\rangle \varphi \iff b \to \langle\!\langle S_1 \rangle\!\rangle \varphi \wedge \neg b \to \langle\!\langle S_2 \rangle\!\rangle \varphi$$
$$\langle \textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2 \varphi \rangle \iff b \to \langle S_1 \rangle \varphi \wedge \neg b \to \langle S_2 \rangle \varphi$$

$$[\![\textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2]\!] \varphi \iff b \to [\![S_1]\!] \varphi \wedge \neg b \to [\![S_2]\!] \varphi$$
$$[\textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2] \varphi \iff b \to [S_1] \varphi \wedge \neg b \to [S_2] \varphi$$

$$\langle\!\langle \textbf{while } b \textbf{ do } S \rangle\!\rangle \varphi \Rightarrow b \to \langle\!\langle S; \textbf{while } b \textbf{ do } S \rangle\!\rangle \varphi \wedge \neg b \to \varphi$$
$$\langle \textbf{while } b \textbf{ do } S \rangle \varphi \Rightarrow b \to \langle S; \textbf{while } b \textbf{ do } S \rangle \varphi \wedge \neg b \to \varphi$$

$$[\![\textbf{while } b \textbf{ do } S]\!] \varphi \Rightarrow b \to [\![S; \textbf{while } b \textbf{ do } S]\!] \varphi \wedge \neg b \to \varphi$$
$$[\textbf{while } b \textbf{ do } S] \varphi \Rightarrow b \to [S; \textbf{while } b \textbf{ do } S] \varphi \wedge \neg b \to \varphi$$

Axioms for parallelism

$$\langle S_1 || S_2 \rangle \varphi \iff \langle S_1 \rangle \langle S_2 \rangle \varphi \vee \langle\!\langle S_1 \rangle\!\rangle \langle \mathbf{cont}_0 || S_2^{up} \rangle \varphi^{up} \vee$$
$$\langle S_2 \rangle \langle S_1 \rangle \varphi \vee \langle\!\langle S_2 \rangle\!\rangle \langle \mathbf{cont}_0 || S_1^{up} \rangle \varphi^{up}$$

$$[S_1 || S_2] \varphi \iff [S_1][S_2] \varphi \wedge [\![S_1]\!] [\mathbf{cont}_0 || S_2^{up}] \varphi^{up} \wedge$$
$$[S_2][S_1] \varphi \wedge [\![S_2]\!] [\mathbf{cont}_0 || S_1^{up}] \varphi^{up}$$

$$\langle\!\langle S_1 || S_2 \rangle\!\rangle \varphi \iff \langle S_1 \rangle \langle\!\langle S_2 \rangle\!\rangle \varphi \vee \langle\!\langle S_1 \rangle\!\rangle \langle\!\langle \mathbf{cont}_0 || S_2^{up} \rangle\!\rangle \varphi^{up} \vee$$
$$\langle S_2 \rangle \langle\!\langle S_1 \rangle\!\rangle \varphi \vee \langle\!\langle S_2 \rangle\!\rangle \langle\!\langle \mathbf{cont}_0 || S_1^{up} \rangle\!\rangle \varphi^{up}$$

$$[\![S_1 || S_2]\!] \varphi \iff [S_1][\![S_2]\!] \varphi \wedge [\![S_1]\!] [\![\mathbf{cont}_0 || S_2^{up}]\!] \varphi^{up} \wedge$$
$$[S_2][\![S_1]\!] \varphi \wedge [\![S_2]\!] [\![\mathbf{cont}_0 || S_1^{up}]\!] \varphi^{up}$$

Related work

- A Sequent Calculus for First-Order Dynamic Logic with Trace Modalities, Bernhard Beckert, Steffen Schlager, Automated Reasoning, 2001
- Dynamic Logic with Trace Semantics, Bernhard Beckert, Daniel Bruns, Automated Deduction - CADE-24, 2013