Reductions for Synthesis Procedures

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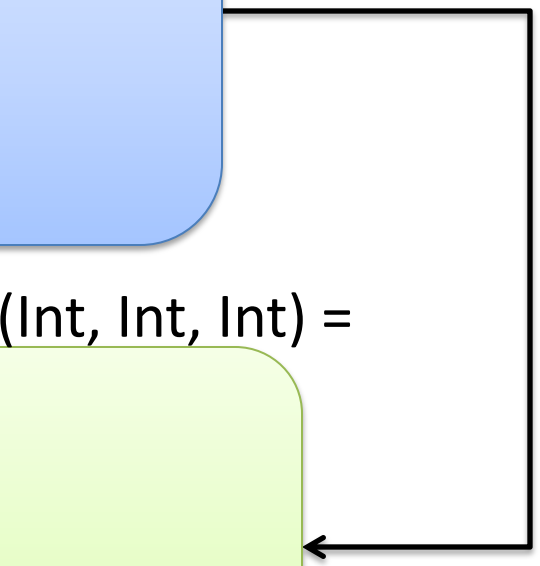
(based on the paper of the same name, to be presented at VMCAI 2013)

Compiling Specifications

```
def secondsToTime(totalSeconds: Int) : (Int, Int, Int) =
  choose { (h: Int, m: Int, s: Int) ⇒
          h * 3600 + m * 60 + s == totalSeconds
     && h ≥ 0
     && m ≥ 0 && m < 60
     && s ≥ 0 && s < 60
  }
```



```
def secondsToTime(totalSeconds: Int) : (Int, Int, Int) =
  val t1 = totalSeconds / 3600
  val t2 = totalSeconds + ((-3600) * t1)
  val t3 = min(t2 / 60, 59)
  val t4 = totalSeconds + ((-3600) * t1) + (-60 * t3)
  (t1, t3, t4)
```

Transforming a relation into a function.

Decision & Synthesis Procedures

For a well-defined class of formulas:

Decision procedure (model-generating)

a theorem prover that always succeeds

- **Input**: a formula
- **Output**: a model of the formula

$5a + 7x = 31$

$\downarrow$

$a \mapsto 2$

$x \mapsto 3$

Synthesis procedure

a synthesizer that always succeeds

- **Input**: a formula, with *input* and *output* variables
- **Output**: a program to compute *output* values from *input* values

Inputs: { a } outputs: { x }

$5a + 7x = 31$

$\downarrow$

$x \mapsto (31 - 5a) / 7$

Synthesis Procedures

- So far, “monolithic” approach; procedures work as black boxes.
- Synthesis for sets is a one-off encoding into integer linear arithmetic.
- Original motivation for current work: general technique to combine synthesis procedures for different theories (*à la* Nelson-Oppen).

Transforming Relations

$$[\![\, \overline{a} \,\langle\, \varphi_1 \,\rangle\, \overline{x} \,]\!] \vdash [\![\, \overline{a} \,\langle\, \varphi_2 \,\rangle\, \overline{x} \,]\!]$$

Input variables

Synthesis predicate

Output variables

$$\forall\, \overline{a}, \overline{x} : \varphi_2 \Rightarrow \varphi_1$$

Refinement

$$\forall\, \overline{a} : (\exists\, \overline{x} : \varphi_1) \Rightarrow (\exists\, \overline{x} : \varphi_2)$$

Domain preservation

Transforming Relations

$$[\![\, \overline{a} \,\langle\, \varphi \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P \mid \overline{T} \,\rangle$$

Input variables

Output variables

Precondition

Program terms

Represents the relation: $P \Rightarrow (\overline{x} = \overline{T})$

$$\forall\, \overline{a} : P \Rightarrow \varphi[\overline{x} \mapsto \overline{T}]$$

Refinement

$$\forall\, \overline{a} : (\exists\, \overline{x} : \varphi) \Rightarrow P$$

Domain preservation

Transforming Relations

Equivalence

$$\frac{[\![\, \overline{a} \langle\, \varphi_1 \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P \mid \overline{T} \,\rangle \qquad \varphi_1 \Leftrightarrow \varphi_2}{[\![\, \overline{a} \langle\, \varphi_2 \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P \mid \overline{T} \,\rangle}$$

Case-Split

$$\frac{[\![\, \overline{a} \langle\, \varphi_1 \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P_1 \mid \overline{T}_1 \,\rangle \qquad [\![\, \overline{a} \langle\, \varphi_2 \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P_2 \mid \overline{T}_2 \,\rangle}{[\![\, \overline{a} \langle\, \varphi_1 \vee \varphi_2 \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P_1 \vee P_2 \mid \textbf{if}(P_1)\ \overline{T}_1\ \textbf{else}\ \overline{T}_2 \,\rangle}$$

One-Point

$$\frac{[\![\, \overline{a} \langle\, \varphi[x_0 \mapsto t] \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P \mid \overline{T} \,\rangle \qquad x_0 \notin \mathrm{vars}(t)}{[\![\, \overline{a} \langle\, x_0 = t \wedge \varphi \,\rangle\, x_0; \overline{x} \,]\!] \vdash \langle\, P \mid \textbf{let}\ \overline{x} := \overline{T}\ \textbf{in}\ t; \overline{x} \,\rangle}$$

Validating Inference Rules

$$\frac{[\![\, \overline{a}\, \langle\, \varphi_1 \,\rangle\, \overline{x}\,]\!] \vdash \langle\, P_1 \mid \overline{T}_1 \,\rangle \qquad [\![\, \overline{a}\, \langle\, \varphi_2 \,\rangle\, \overline{x}\,]\!] \vdash \langle\, P_2 \mid \overline{T}_2 \,\rangle}{[\![\, \overline{a}\, \langle\, \varphi_1 \vee \varphi_2 \,\rangle\, \overline{x}\,]\!] \vdash \langle\, P_1 \vee P_2 \mid \textbf{if}(P_1)\ \overline{T}_1\ \textbf{else}\ \overline{T}_2 \,\rangle}$$

Refinement $\forall\, \overline{a} : P \Rightarrow \varphi[\overline{x} \mapsto \overline{T}]$

$$P_1 \vee P_2 \Rightarrow (\varphi_1 \vee \varphi_2)[\overline{x} \mapsto (\textbf{if}(P_1)\ \overline{T}_1\ \textbf{else}\ \overline{T}_2)]$$

Assuming P_1

$$(\varphi_1 \vee \varphi_2)[\overline{x} \mapsto \overline{T}_1] \longrightarrow \varphi_1[\overline{x} \mapsto \overline{T}_1] \vee \varphi_2[\overline{x} \mapsto \overline{T}_1]$$

Domain preservation $\forall\, \overline{a} : (\exists\, \overline{x} : \varphi) \Rightarrow P$

$$(\exists\, \overline{x} : \varphi_1 \vee \varphi_2) \Rightarrow P_1 \vee P_2$$

$$(\exists\, \overline{x} : \varphi_1) \vee (\exists\, \overline{x} : \varphi_2) \Rightarrow P_1 \vee P_2$$

Advantages of Rule-based Synthesis

- Simplifies correctness arguments.
- Simple to reason about interleavings of synthesis steps for different theories.
- Partial synthesis:
  - Synthesis predicates partially in unsupported logics.
  - Program size / efficiency tradeoffs.

Relation to Quantifier Elimination

- A problem of the form:

$$[\![\, \overline{a} \,\langle\, \varphi \,\rangle\, \overline{x} \,]\!]$$

- Corresponds to constructively solving the **quantifier elimination** problem:

$$\exists\, \overline{x} : \varphi(\, \overline{a}, \overline{x}\,)$$

- In the solution, P corresponds to the result of Q.E. and $\overline{T}$ are *witness terms.*

$$\langle\, P \mid \overline{T} \,\rangle$$

Synthesis Procedures: Applicability

- In principle, any model-generating decision procedure can be adapted; at the limit, the generated program *is* the decision procedure.

- Synthesis procedures derived from Q.E.:
  - Integer linear (Presburger) arithmetic
  - Sets with cardinalities
  - Algebraic data types (lists, trees, etc.)

Synthesis for Linear Integer Arithmetic

$$\frac{[\![\, a \,\langle\, 7t \le a \wedge 5a \le 12t \,\rangle\, t \,]\!] \vdash \langle\, \lceil 5a/12 \rceil \le \lfloor 3a/7 \rfloor \mid \lceil 5a/12 \rceil \,\rangle}{[\![\, a \,\langle\, 5x + 7y = a \wedge 0 \le x \wedge x \le y \,\rangle\, x,y \,]\!] \vdash \langle\, \lceil 5a/12 \rceil \le \lfloor 3a/7 \rfloor \mid \textbf{let } t = \lceil 5a/12 \rceil \textbf{ in } (-7t+3a,\ 5t-2a) \,\rangle}$$

$5x + 7y = a$

One-dimensional solution space.

$x = -7t + 3a$

$y = 5t - 2a$

is a solution for any t.

$7t \le a \wedge 5a \le 12t$

t is bound on both sides, and admits a solution whenever

$\lceil 5a/12 \rceil \le \lfloor 3a/7 \rfloor$

Synthesis for Term Algebras

List ::= Nil | Cons(h : Int, t : List)

Unification-1
$$\frac{[\![\, \overline{a} \,\langle\, t_1 = t_3 \wedge t_2 = t_4 \wedge \varphi \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P \mid \overline{T} \,\rangle}{[\![\, \overline{a} \,\langle\, \mathrm{Cons}(t_1, t_2) = \mathrm{Cons}(t_3, t_4) \wedge \varphi \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P \mid \overline{T} \,\rangle}$$

Unification-2
$$[\![\, \overline{a} \,\langle\, \mathrm{Cons}(t_1, t_2) = \mathrm{Nil} \wedge \varphi \,\rangle\, \overline{x} \,]\!] \vdash \langle\, \textbf{false} \mid \_ \,\rangle$$

Unification-3 ...

Dual
$$\frac{[\![\, \overline{a} \,\langle\, t_1 = a.h \wedge t_2 = a.t \wedge \varphi \,\rangle\, \overline{x} \,]\!] \vdash \langle\, P \mid \overline{T} \,\rangle}{[\![\, \overline{a} \,\langle\, a = \mathrm{Cons}(t_1, t_2) \wedge \varphi \,\rangle\, \overline{x} \,]\!] \vdash \langle\, \mathrm{isCons}(a) \wedge P \mid \overline{T} \,\rangle}$$

Synthesis for Term Algebras

$[\![\ a\ \langle\ x \neq a\ \rangle\ x,y\ ]\!] \vdash$
$\langle$ **true** | (a+1, Nil) $\rangle$

$[\![\ b\ \langle\ b \neq y\ \rangle\ x,y\ ]\!] \vdash$
$\langle$ **true** | (0, Δ(b)) $\rangle$

$[\![\ a,b\ \langle\ \text{Cons}(x,b) \neq \text{Cons}(a,y)\ \rangle\ x,y\ ]\!] \vdash$
$\langle$ **true** | **if**(**true**) (a+1, Nil) **else** (0 , Δ(b)) $\rangle$

Disequalities between constructors are trivially true or decomposable.

$$x \neq a \vee b \neq y$$

Proceed with case-split.

Given a conjunction of disequalities:

$$x \neq T_1 \wedge \ldots \wedge x \neq T_n$$

We can always solve for *x*. We call *Δ* a computable function s.t.

$$\Delta(T_1,\ldots,T_n) \neq T_i$$

(Alternative Case-Splitting)

Case-Split
$$\frac{[\![\, \bar{a} \langle\, \varphi_1 \,\rangle \bar{x} \,]\!] \vdash \langle\, P_1 \mid \bar{T}_1 \,\rangle \qquad [\![\, \bar{a} \langle\, \varphi_2 \,\rangle \bar{x} \,]\!] \vdash \langle\, P_2 \mid \bar{T}_2 \,\rangle}{[\![\, \bar{a} \langle\, \varphi_1 \vee \varphi_2 \,\rangle \bar{x} \,]\!] \vdash \langle\, P_1 \vee P_2 \mid \textbf{if}(P_1)\, \bar{T}_1 \,\textbf{else}\, \bar{T}_2 \,\rangle}$$

$$\frac{[\![\, \bar{a} \langle\, \varphi_1 \,\rangle \bar{x} \,]\!] \vdash \langle\, P_1 \mid \bar{T}_1 \,\rangle \qquad [\![\, \bar{a} \langle\, \neg P_1 \wedge \varphi_2 \,\rangle \bar{x} \,]\!] \vdash \langle\, P_2 \mid \bar{T}_2 \,\rangle}{[\![\, \bar{a} \langle\, \varphi_1 \vee \varphi_2 \,\rangle \bar{x} \,]\!] \vdash \langle\, P_1 \vee P_2 \mid \textbf{if}(P_1)\, \bar{T}_1 \,\textbf{else}\, \bar{T}_2 \,\rangle}$$

May simplify the problem, but requires closing a branch first.

Synthesis for Arrays

- How do you handle $[\![a \langle \text{a[x]} = \text{x} \rangle x]\!]$?
- In general, need to generate a search over indices.
- Overall strategy:
  - Project φ on the index variables. Use as a filter for the search, or use an *enumerating* synthesis procedure.
  - Within the loop, the problem is purely in the element theory.

Implementation(s) Status

- *Comfusy*
  - Synthesis for (conjunctions in) integer linear arithmetic and sets with cardinality constraints.
  - Implemented as a compiler plugin for Scala.

Mayer, Kuncak, Piskac, Suter, "*Complete Functional Synthesis*", PLDI 2010.

Spielmann, Kuncak "*Synthesis for Unbounded Bit-Vector Arithmetic",* IJCAR 2012.

- Ongoing work:
  - System based on inference rules.
  - Mixture of *definite* rules and *heuristics* to handle problems that need more than straight-line code.

Jacobs, Kuncak, Suter, "*Reductions for Synthesis Procedures*", VMCAI 2013.

Reductions for Synthesis Procedures

- Synthesis: rewrite a relation into a function.
  - Synthesis procedures: algorithms that always succeed for a class of problems.

- Rule-based framework for designing synthesis procedures.
  - Allows for interleavings of theory steps.
  - Sets of rules for term algebras, linear arithmetic, arrays.

Thank you.